半正定値計画を用いた距離正則グラフのユークリッド歪みの評価

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Evaluation of the Euclidean distortion of a distance-regular graph using semidefinite programing method

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Let X be a metric space and F be an embedding from X to the ℓ_2-Hilbert space. The distortion of F is defined by the product of the Lipschitz constant of F and the Lipschitz constant of F^{-1}, and the Euclidean distortion of X is defined by the infimum of distortion amongst the embedding of X. It is not easy to determine the Euclidean distortion.
There are many researches of the Euclidean distortions of finite graphs. Moreover, for distance-regular graphs, lower bounds for the Euclidean distortions are known. In this paper, using semidefinite programing method, we rediscover lower bounds of Linial, London and Rabinovich.

Key words: distance-regular graph; Euclidean distortion; semidefinite programming method.

1 Introduction

Let (X, d_X) and (Y, d_Y) be metric spaces, and F be a map from (X, d_X) into (Y, d_Y). For a positive real C, a map F is called a *C-bilipschitz embedding* if there exists $r > 0$ such that

$$r d_X(x, y) \leq d_Y(F(x), F(y)) \leq rC d_X(x, y) \qquad (1)$$

for any $x, y \in X$. A *bilipschitz embedding* is an embedding which is C-bilipschitz for some $C \in \mathbb{R}_{\geq 0}$. Remark that if F is a bilipschitz embedding then F is injection. The smallest constant C for which there exists $r > 0$ such that (1) is satisfied is called the distortion of F. (It is easy to see that such smallest constant exists.) The infimum of distortions of bilipschitz embeddings of X into Y is denoted $c_Y(X)$.

The distortion of F is equal to

$$\frac{\sup_{\substack{x,y\in X\\ x\neq y}} \frac{d_Y(F(x),F(y))}{d_X(x,y)}}{\inf_{\substack{x,y\in X\\ x\neq y}} \frac{d_Y(F(x),F(y))}{d_X(x,y)}}.$$

The numerator of the above is called the Lipschitz constant of F and the reciprocal of the denominator is called the Lipschitz constant of F^{-1}. By the definition of distortion, the distortion of F is at least 1 and if the distortion of F is equal to 1 then F is an isometry. Thus $c_Y(X) \geq 1$ and if $c_Y(X) = 1$ then X and Y are isometric.

When $Y = \ell_p$ we use the notation $c_Y(\cdot) = c_p(\cdot)$ and call this number the *ℓ_p-distortion of X*. The parameter $c_2(X)$ is called the *Euclidean distortion of X*. We can obtain a trivial upper bound for $c_p(\cdot)$ of a finite graph. For a finite graph $\Gamma = (X, E)$ with diameter d, let $F\colon \Gamma \to \mathbb{R}^X$, $x \mapsto e_x$ (standard representation) as $\ell_p^{|X|}$, then $\|F(x) - F(y)\|_p = 2^{1/p}(1 - \delta_{xy})$, where δ_{xy} is the Kronecker delta. Hence the distortion of F is d. This implies $c_p(\Gamma) \leq d$.

Let X be an n-points metric space. Bourgain[2] showed that $c_2(X) = O(\log n)$. However, it is not easy to determine the exact value of the Euclidean distortion $c_2(X)$ of a given n-point metric space. We have few examples of finite metric spaces whose Euclidean distortion is exactly given. The list of the examples only includes Hamming graphs (due to Enflo[4]), Johnson graphs and all strongly regular graphs (due to Vallentin[7]). Linial, London and Rabinovich[5] gave an algorithm to find the Euclidean distortions of finite metric spaces, and Linial and Magen[6] showed some properties of the optimal embedding for the Euclidean distortions. However it is not easy to determine the Euclidean distortion of given metric space.

Our aim is to find a "good" evaluation of the Euclidean distortion of a finite graph Γ of diameter d. We already have the trivial evaluation:

$$1 \leq c_2(\Gamma) \leq d.$$

Our concern is sharper evaluation of $c_2(\Gamma)$. The results of Linial, London and Rabinovich[5] also give some lower bounds for the Euclidean distortions. After their work, Vallentin[7] showed specific lower bounds for the Euclidean distortions of distance-regular graphs using the parameters of the graphs.

Theorem 1.1 (Vallentin[7]). *Let Γ be a distance-regular graph with d, $\{\theta_h\}_{h=0}^d$ be the eigenvalues of Γ and $\{v_i\}_{i=0}^d$ be the polynomials related to Γ. Then the Euclidean distortion $c_2(\Gamma)$ of Γ have the following lower bound:*

$$c_2(\Gamma)^2 \geq \frac{d^2 v_d(\theta_0)}{v_1(\theta_0)} \min_{1\leq h\leq d} \frac{v_1(\theta_0) - v_1(\theta_h)}{v_d(\theta_0) - v_d(\theta_h)}.$$

The definition of distance-regular graph and the notation appeared in Theorem 1.1 are explained in Section 2.1.

In this paper we first rediscover the results of Linial, London and Rabinovich[5] and Linial and Magen[6] using semidefinite programing method. Next we refine Vallentin's lower bound. Finally we give upper bounds for the Euclidean distortions of distance-regular graphs using the primitive idempotents of the graphs.

2 Preliminary

Throughout this paper, for a set X, let $\mathrm{Mat}_{\mathbb{R}}(X)$ be the set of matrices of size $|X|$ over $\mathbb{R}$ such that the rows and the columns are indexed by X, and $\mathrm{Sym}_{\mathbb{R}}(X)$ be the set of symmetric matrices in $\mathrm{Mat}_{\mathbb{R}}(X)$. For $M \in \mathrm{Mat}_{\mathbb{R}}(X)$ and $x, y \in X$, M_{xy} denotes the (x,y)-entry of M.

2.1 Distance-regular graph

We consider only finite undirected graphs without loops or multiple edges. Let $\Gamma = (X, E)$ be such a graph, where X and E are the vertex and edge sets. For two vertices x and y, $\partial_\Gamma(x,y)$ denotes the length of the shortest path joining x and y. The *diameter* of Γ is the maximal distance occurring in Γ. A connected graph Γ with diameter d is *distace-reguler* if each of the following numbers

$$|\{z \in X \mid \partial_\Gamma(x,z) = i-1,\ \partial_\Gamma(z,y) = 1\}| \qquad (2)$$
$$|\{z \in X \mid \partial_\Gamma(x,z) = i,\ \partial_\Gamma(z,y) = 1\}| \qquad (3)$$
$$|\{z \in X \mid \partial_\Gamma(x,z) = i+1,\ \partial_\Gamma(z,y) = 1\}| \qquad (4)$$

$(i = 0, 1, \ldots, d)$ does not depend on the choice of $x, y \in X$ with $\partial_\Gamma(x,y) = i$. The numbers (2), (3) and (4) are denoted by b_i, a_i and c_i, respectively. We always assume that $c_0 = d_d = 0$. The constant b_0 is called the *valency* of Γ and is denoted by k. Remark that $a_i + b_i + c_i = k$ for each $i = 0, 1, \ldots, d$. The numbers a_i, b_i, c_i are called the *intersection numbers* and the array $\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$ is called the *intersection array* of Γ.

Let $\Gamma = (X, E)$ be a distance-regular graph with diameter d and $|X| = n$. For $i = 0, 1, \ldots, d$, let A_i be the matrix in $\mathrm{Mat}_{\mathbb{R}}(X)$ and the (x,y)-entry is 1 whenever x and y are at distance i and 0 otherwise. We call A_i the *i-th distance matrix* of Γ. We abbreviate $A := A_1$ and call this the *adjacency matrix* of Γ. We have the recurrence relation

$$A_i A = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1}.$$

This recurrence relation implies that there exist polynomials $v_i(\theta)$ of degree exactly i such that $A_i = v_i(A)$. Note $v_0(\theta) = 1$ and $v_1(\theta) = \theta$. We find that $A_0, A_1, \ldots, A_d$ form a basis for a commutative algebra $\mathfrak{A} = \mathrm{Span}_{\mathbb{R}}\{A_i\}_{i=0}^d \subset \mathrm{Sym}_{\mathbb{R}}(X)$. We call $\mathfrak{A}$ the *Bose-Mesner algebra* of Γ. Since $A_i = v_i(A)$, it turns out that A generates $\mathfrak{A}$. By[3, p.45], $\mathfrak{A}$ has a second basis $E_0, E_1, \ldots, E_d$ of the primitive idempotents of Γ, and A can be written as $A = \sum_{h=0}^d \theta_h E_h$, where θ_h is the eigenvalue of A associated with E_h. Remark that $\{E_h\}_{h=0}^d$ are positive semi-definite. We denote by m_h the multiplicity of θ_h. For an eigenvalue $\theta = \theta_h$ we will also write E_θ instead of E_h.

Remark 2.1. *Take $M \in \mathfrak{A}$. Since $\{E_h\}_{h=0}^d$ are positive semi-definite, M is positive semi-definite if and only if there exist non-negative numbers $a_0, a_1, \ldots a_d$ such that $M = \sum_{h=0}^d a_h E_h$.*

Since $A_i = v_i(A)$ and E_h are idempotents, we have

$$A_i = \sum_{h=0}^{d} v_i(\theta_h) E_h \qquad (5)$$

for $i \in \{0, 1, \ldots, d\}$. Let P be the square matrix of size $d+1$ whose the (h,i)-entry is $v_i(\theta_h)$, that is,

$$P = \begin{pmatrix} v_0(\theta_0) & v_1(\theta_0) & \cdots & v_d(\theta_0) \\ v_0(\theta_1) & v_1(\theta_1) & \cdots & v_d(\theta_1) \\ \vdots & \vdots & \ddots & \vdots \\ v_0(\theta_d) & v_1(\theta_d) & \cdots & v_d(\theta_d) \end{pmatrix}.$$

Then (5) implies that P is non-singular. Hence there exists a square matrix $Q = (Q_h(i))_{i,h=0}^d$ of size $d+1$ such that

$$E_h = \frac{1}{n} \sum_{i=0}^{d} Q_h(i) A_i,$$

that is,

$$\sum_{h=0}^{d} Q_h(k) v_l(\theta_h) = n\delta_{kl} \qquad (6)$$

for $k, l \in \{0, 1, \ldots, d\}$. Is is known that

$$Q_0(i) = 1 \qquad (i \in \{0, 1, \ldots, d\}), \qquad (7)$$

and

$$\frac{v_i(\theta_h)}{v_i(\theta_0)} = \frac{Q_h(i)}{Q_h(0)} \qquad (8)$$

for each $h, i \in \{0, 1, \ldots d\}$. The reader is referred to Bannai–Ito[1] for more properties of P and Q.

Remark 2.2. *For $M = \sum_{h=0}^d a_h E_h$ and $x, y \in X$ with $\partial_\Gamma(x,y) = i$, we have*

$$\begin{aligned} M_{xy} &= \left(\sum_{h=0}^{d} a_h E_h \right)_{xy} \\ &= \left(\sum_{h=0}^{d} a_h \frac{1}{n} \sum_{i=0}^{d} Q_h(i) A_i \right)_{xy} \\ &= \frac{1}{n} \sum_{h=0}^{d} a_h Q_h(i). \end{aligned}$$

2.2 Embedding of graphs onto spheres

Definition 2.3. *Let (X, d_X) be a finite metric space and (Y, d_Y) be a metric space. We say that an embedding $F: X \to Y$ is* faithful *if for every two pairs $(x, y), (x', y') \in X \times X$ we have*

$$d_Y(F(x), F(y)) = d_Y(F(x'), F(y'))$$

whenever $d_X(x, y) = d_X(x', y')$.

Suppose $(X, d_X) = (\Gamma, \partial_\Gamma)$ and F is faithful, then we write $d_Y(F(x), F(y)) = d_Y(i)$ whenever $\partial_\Gamma(x, y) = i$.

Definition 2.4. *Let (X, d_X) and (Y, d_Y) be metric spaces, and let F be an embedding from X onto Y.*

- *A pair $(x, y) \in X \times X$ is called* expanded *if $d_Y(F(x), F(y))/d_X(x, y)$ is maximal among all pairs in $X \times X$.*
- *A pair $(x, y) \in X \times X$ is called* contracted *if $d_Y(F(x), F(y))/d_X(x, y)$ is minimal among all pairs in $X \times X$.*

Moreover suppose $(X, d_X) = (\Gamma, \partial_\Gamma)$ and F is faithful.

- *A distance i is called* expanded *if there exists an expanded pair (x, y) such that $\partial_\Gamma(x, y) = i$.*
- *A distance i is called* contacted *if there exists a contracted pair (x, y) such that $\partial_\Gamma(x, y) = i$.*

Lemma 2.5. *Let $\Gamma = (X, E)$ be a graph of diameter d, and F be a faithful embedding Γ onto a sphere $\mathbb{S}^N$. Let ∂ be the distance among elements of $F(\Gamma)$ on $\mathbb{S}^N$. Then the expanded distance is only 1, i.e., $\partial(1)/1 > \partial(k)/k$ for $k \in \{2, 3, \ldots, d\}$.*

Proof. For $k \in \{2, 3, \ldots, d\}$, there exists a path $x_0 \sim x_1 \sim x_2 \sim \cdots \sim x_k$ such that $\partial_\Gamma(x_0, x_k) = k$ and $\partial_\Gamma(x_{i-1}, x_i) = 1$ for $i \in \{1, 2, \ldots, k\}$. By the triangle inequality, we have

$$\begin{aligned}\partial(k) &= \partial(F(x_0), F(x_k)) \\ &\leq \sum_{i=1}^{k} \partial(F(x_{i-1}), F(x_i)) = \sum_{i=1}^{k} \partial(1) = k\partial(1),\end{aligned}$$

i.e., $\partial(1)/1 \geq \partial(k)/k$ follows.

Assume $\partial(1)/1 = \partial(k)/k$, then

$$\partial(F(x_0), F(x_k)) = \sum_{i=1}^{k} \partial(F(x_{i-1}), F(x_i))$$

holds. This implies that $F(x_0), F(x_{k-1}), F(x_k)$ lie on a line. However, this contradicts that $F(x_0), F(x_{k-1}), F(x_k)$ lie on $\mathbb{S}^N$. Hence we get $\partial(1)/1 > \partial(k)/k$. □

For a finite subset Z in a Euclidean space, the *Gram matrix* G of Z in $\mathrm{Sym}_{\mathbb{R}}(Z)$ is defined by $G_{\boldsymbol{x}\boldsymbol{y}} = \boldsymbol{x} \cdot \boldsymbol{y}$ for $\boldsymbol{x}, \boldsymbol{y} \in Z$, where $\cdot$ is the standard inner product of the Euclidean space.

Lemma 2.6. *Let $\Gamma = (X, E)$ be a graph. For a semi-definite matrix M in $\mathrm{Sym}_{\mathbb{R}}(X)$, there exists an embedding of Γ into the Euclidean space of dimension $\mathrm{Rank}\, M$.*

Proof. Since M is semi-definite, there exists $(\mathrm{Rank}\, M \times |X|)$-matrix B such that $B^T B = M$. Then we can regard the column vectors of B as vectors in $\mathbb{R}^{\mathrm{Rank}\, M}$. □

Let $\Gamma = (X, E)$ be a distance-regular graph with d, $\{\theta_h\}_{h=0}^d$ be the eigenvalues of Γ and $\{E_h\}_{h=0}^d$ be the primitive idempotents of Γ. For E_h, put $m = \mathrm{Rank}\, E_h$. We define the map F_h from X into a Euclidean space such that, for $x \in X$, $F_h(x)$ is the x-th column of E_h. Since E_h is an idempotent, $E_h^T E_h = E_h$ holds. This implies the Gram matrix of $F_h(\Gamma)$ is E_h. By Lemma 2.6, F_h is embedding of Γ into $\mathbb{R}^m$. Moreover

$$F_h(x) \cdot F_h(y) = (E_h)_{xy} = \frac{1}{n} Q_h(i)$$

for $x, y \in X$ with $\partial_\Gamma(x, y) = i$. Note that for each $x \in X$, we have $\|F_h(x)\|^2 = Q_h(0)/n$, hence $F_h(\Gamma)$ is on a sphere $\mathbb{S}^{m-1}$. On the other hand we have

$$\|F_h(x), F_h(y))\|^2 = (Q_h(0) - Q_h(i))/n \qquad (9)$$

for $x, y \in X$ with $\partial_\Gamma(x, y) = i$, hence F_h is faithful. By Lemma 2.5, the expanded distance is only 1. Therefore we get the following

Lemma 2.7. *F_h is embedding of Γ onto $\mathbb{S}^{m-1}$ such that the expanded distance is only 1.*

3 Semi-definite programing

For a finite graph $\Gamma = (X, E)$ with the path distance ∂_Γ, let $\check{\mathcal{C}}$ be the set of (M, r^2) satisfying $(M, r^2) \neq (0, 0)$, $M \in \mathrm{Sym}_{\mathbb{R}}(X)$ is positive semi-definite, $r \geq 0$ and

$$M_{xx} + M_{yy} - 2M_{xy} - r^2 \partial_\Gamma(x, y)^2 \geq 0$$

for $x, y \in X$ $(x \neq y)$ and, for a non-negative real D, let $\check{\mathcal{S}_D}$ be the set of (M, r^2) satisfying $(M, r^2) \neq (0, 0)$, $M \in \mathrm{Sym}_{\mathbb{R}}(X)$ and

$$-M_{xx} - M_{yy} + 2M_{xy} + r^2 D \partial_\Gamma(x, y)^2 \geq 0$$

for $x, y \in X$ $(x \neq y)$. Then $\check{\mathcal{C}}$ and $\check{\mathcal{S}_D}$ are cones in $\mathrm{Sym}_{\mathbb{R}}(X) \oplus \mathbb{R}$.

Lemma 3.1. *There exists a $\sqrt{D}$-bilipschitz embedding of Γ into ℓ_2 if and only if $\check{\mathcal{C}} \cap \check{\mathcal{S}_D} \neq \emptyset$.*

Proof. Assume F is a $\sqrt{D}$-bilipschitz embedding of Γ into ℓ_2. Let G be the Gram matrix of $F(X)$. Then $\|F(x) - F(y)\|^2 = G_{xx} + G_{yy} - 2G_{xy}$. By the definition of bilipschitz, we can check $(G, r^2) \in \check{\mathcal{C}}$ and $(G, r^2) \in \check{\mathcal{S}_D}$ for some $r > 0$. Hence this implies $\check{\mathcal{C}} \cap \check{\mathcal{S}_D} \neq \emptyset$.

Assume $(M, r^2) \in \check{\mathcal{C}} \cap \check{\mathcal{S}_D}$. By Lemma 2.6, there exists an embedding F of Γ into the Euclidean space with

$$\|F(x) - F(y)\|^2 = M_{xx} + M_{yy} - 2M_{xy}. \qquad (10)$$

Then F satisfies the condition of $\sqrt{D}$-bilipschitz. □

Lemma 3.1 implies that C is the Euclidean distortion of Γ if and only if $C = \inf\{\sqrt{D} \mid D > 0 \text{ with } \check{\mathcal{C}} \cap \check{\mathcal{S}_D} \neq \emptyset\}$. The study of the intersection of two cones $\check{\mathcal{C}}$ and $\check{\mathcal{S}_D}$ using convex analysis is called *positive semi-definite programming method.* If Γ is distance-regular, the two cones $\check{\mathcal{C}}$ and $\check{\mathcal{S}_D}$ became more simple.

Theorem 3.2 (Vallentin[7]). *Let $\Gamma = (X, E)$ be a distance-regular graph. Then, there exists a faithful embedding of Γ into Euclidean space with minimal distortion. In particular, this embedding is in a sphere centered at the origin in the Euclidean space.*

Theorem 3.2 yields that for Γ is distance-regular, an optimal solution $(M, r^2) \in \check{\mathcal{C}} \cap \check{\mathcal{S}_D}$ of $\inf\{\sqrt{D} \mid D > 0 \text{ with } \check{\mathcal{C}} \cap \check{\mathcal{S}_D} \neq \emptyset\}$ can be found in $\mathfrak{A} \oplus \mathbb{R}$. By Remarks 2.1 and 2.2, $\mathcal{C} := \check{\mathcal{C}} \cap (\mathfrak{A} \oplus \mathbb{R})$ can be regarded as a cone

$$\left\{ (a_0, \ldots, a_d, R) \;\middle|\; \begin{array}{l} a_h \geq 0 \quad (0 \leq h \leq d), \\ R \geq 0, \\ \frac{2}{n}\sum_{h=0}^{d} a_h(Q_h(0) - Q_h(i)) \\ -Ri^2 \geq 0 \qquad (1 \leq i \leq d), \\ (a_0, \ldots, a_d, R) \neq \mathbf{0} \end{array} \right\}$$

in $\mathbb{R}^{d+2}$ and also $\mathcal{S}_D := \check{\mathcal{S}_D} \cap (\mathfrak{A} \oplus \mathbb{R})$ can be regarded as a cone

$$\left\{ (a_0, \ldots, a_d, R) \;\middle|\; \begin{array}{l} a_h \in \mathbb{R} \quad (0 \leq h \leq d),\ R \in \mathbb{R}, \\ -\frac{2}{n}\sum_{h=0}^{d} a_h(Q_h(0) - Q_h(i)) \\ +Ri^2 D \geq 0 \qquad (1 \leq i \leq d), \\ (a_0, \ldots, a_d, R) \neq \mathbf{0} \end{array} \right\}.$$

Henceforth, we write $(\boldsymbol{a}, R)$ or $((a_h)_{h=0}^d, R)$ instead of $(a_0, \ldots, a_d, R)$. For $(\boldsymbol{a}, R), (\boldsymbol{a}', R') \in \mathbb{R}^{d+2}$, the standard inner product is given by $(\boldsymbol{a}, R) \cdot (\boldsymbol{a}', R') = \sum_{h=0}^{d} a_h a'_h + RR'$. The *dual cones of $\mathcal{C}$ and $\mathcal{S}_D$* are defined by

$$\mathcal{C}^* = \{(\boldsymbol{b}, s) \in \mathbb{R}^{d+2} \mid (\boldsymbol{a}, r) \cdot (\boldsymbol{b}, s) \geq 0,\ (\boldsymbol{a}, r) \in \mathcal{C}\}$$

and

$$\mathcal{S}_D^* = \{(\boldsymbol{b}, s) \in \mathbb{R}^{d+2} \mid (\boldsymbol{a}, r) \cdot (\boldsymbol{b}, s) \geq 0,\ (\boldsymbol{a}, r) \in \mathcal{S}_D\},$$

respectively. Then $\mathcal{C}^*$ is written in

$$\{(\boldsymbol{b}, s) \mid b_h \geq 0\ (0 \leq h \leq d),\ s \geq 0\} + \left\{ \sum_{i=1}^{d} \alpha_i \left(\left(\tfrac{2}{n}(Q_h(0) - Q_h(i))\right)_{h=0}^{d}, -i^2 \right) \;\middle|\; \alpha_i \geq 0 \right\}, \qquad (11)$$

and also $\mathcal{S}_D^*$ is

$$\left\{ \sum_{i=1}^{d} \beta_i \left(\left(-\tfrac{2}{n}(Q_h(0) - Q_h(i))\right)_{h=0}^{d}, i^2 D \right) \;\middle|\; \beta_i \geq 0 \right\}. \qquad (12)$$

Lemma 3.3. *For two nonnegative reals D and D', if $D \leq D'$, then $\mathcal{S}_D \subset \mathcal{S}_{D'}$.*

Lemma 3.4. *If $\mathcal{C} \cap \mathcal{S}_D \neq \emptyset$, then $\mathcal{C}^* \cap (-\mathcal{S}_D^*)^\circ = \emptyset$, where $(-\mathcal{S}_D^*)^\circ$ is the interior of $-\mathcal{S}_D^*$.*

Proof. Assume $\mathcal{C}^* \cap (-\mathcal{S}_D^*)^\circ \neq \emptyset$. Let $(\boldsymbol{a}, R) \in \mathcal{C} \cap \mathcal{S}_D$ and $(\boldsymbol{b}, s) \in \mathcal{C}^* \cap (-\mathcal{S}_D^*)^\circ$. Since $(\boldsymbol{a}, R) \in \mathcal{C}$ and $(\boldsymbol{b}, s) \in \mathcal{C}^*$, we have $(\boldsymbol{a}, R) \cdot (\boldsymbol{b}, s) \geq 0$. On the other hand, by $(\boldsymbol{a}, R) \in \mathcal{S}_D$ and $(\boldsymbol{b}, s) \in (-\mathcal{S}_D^*)^\circ$, we have $(\boldsymbol{a}, R) \cdot (\boldsymbol{b}, s) < 0$. It is contradiction. □

Theorem 3.5 (SDP method). *If $D \in \mathbb{R}$ satisfies $\mathcal{C}^* \cap (-\mathcal{S}_D^*) \neq \emptyset$, then $D \leq c_2(\Gamma)^2$ holds.*

Proof. By Lemmas 3.1 and 3.3, we have $\mathcal{C} \cap \mathcal{S}_C \neq \emptyset$ for any C with $C \geq c_2(\Gamma)^2$. This implies if $\mathcal{C} \cap \mathcal{S}_D = \emptyset$, then $D < c_2(\Gamma)^2$. By Lemma 3.4, if $\mathcal{C}^* \cap (-\mathcal{S}_D^*)^\circ \neq \emptyset$, then $D < c_2(\Gamma)$. Moreover if $\overline{\mathcal{C}^* \cap (-\mathcal{S}_D^*)^\circ} = \mathcal{C}^* \cap (-\mathcal{S}_D^*) \neq \emptyset$, then $D \leq c_2(\Gamma)$ also. □

Using Theorem 3.5, we find the lower bound for $c_2(\Gamma)$. Fix nonnegative reals $\boldsymbol{b} = (b_0, b_1, \ldots, b_d)$. Assume $\mathcal{C}^* \cap (-\mathcal{S}_D^*) \neq \emptyset$, then there exist positive reals α_i, β_i and s such that

$$b_h = \frac{2}{n}\sum_{i=1}^{d}(\beta_i - \alpha_i)(Q_h(0) - Q_h(i)) \qquad (13)$$

for $h \in \{0, 1, \ldots, d\}$ and

$$s - \sum_{i=1}^{d} \alpha_i i^2 = -\sum_{i=1}^{d} \beta_i i^2 D \qquad (14)$$

by (11) and (12).

Remark 3.6. *Under the assumption, we have $b_0 = 0$. Because $b_0 = \frac{2}{n}\sum_{i=1}^{d}(\beta_i - \alpha_i)(Q_0(0) - Q_0(i)) = 0$ by (7).*

Let $\hat{Q} = (Q_h(0) - Q_h(i))_{i,h=1}^{d}$ and $\hat{P} = -\frac{1}{n}(v_i(\theta_h))_{h,i=1}^{d}$.

Lemma 3.7. $\hat{Q}^{-1} = \hat{P}$.

Proof. (6) and (7) imply that, for $k, l \in \{1, 2, \ldots, d\}$,

$$\begin{aligned}(\hat{Q}\hat{P})_{k,l} &= -\frac{1}{n}\sum_{h=1}^{d}(Q_h(0) - Q_h(k))v_l(\theta_h)\\ &= -\frac{1}{n}\sum_{h=1}^{d}Q_h(0)v_l(\theta_h) + \frac{1}{n}\sum_{h=1}^{d}Q_h(k)v_l(\theta_h)\\ &= -\frac{1}{n}(0 - Q_0(0)v_l(\theta_0))\\ &\quad + \frac{1}{n}(n\delta_{k,l} - Q_0(k)v_l(\theta_0))\\ &= \frac{1}{n}v_l(\theta_0) + \delta_{k,l} - \frac{1}{n}v_l(\theta_0)\\ &= \delta_{k,l}.\end{aligned}$$

Hence the desired result holds. □

By (13), we have

$$(b_h)_{h=1}^{d} = \frac{2}{n}(\beta_i - \alpha_i)_{i=1}^{d}\hat{Q}.$$

By Lemma 3.7, we have

$$(\beta_i - \alpha_i)_{i=1}^{d} = \frac{n}{2}(b_h)_{h=1}^{d}\hat{P}.$$

Hence, for $i \in \{1, 2, \ldots, d\}$,

$$\beta_i - \alpha_i = -\frac{1}{2}\sum_{h=1}^{d} b_h v_i(\theta_h).$$

Since $\alpha_i = \beta_i + \frac{1}{2}\sum_{h=1}^{d} b_h v_i(\theta_h) \geq 0$ and $\beta_i \geq 0$, we have

$$\beta_i \geq \max\{0, -\frac{1}{2}\sum_{h=1}^{d} b_h v_i(\theta_h)\}. \tag{15}$$

Put $\gamma_i := \frac{1}{2}\sum_{h=1}^{d} b_h v_i(\theta_h)$ and $\gamma^{\pm} := \{i \mid \pm\gamma_i > 0\}$.

By (14), we have

$$D = \frac{\sum_{i=1}^{d}\alpha_i i^2 - s}{\sum_{i=1}^{d}\beta_i i^2}.$$

Then

$$\begin{aligned}D &= \frac{\sum_{i=1}^{d}(\beta_i + \gamma_i)i^2 - s}{\sum_{i=1}^{d}\beta_i i^2}\\ &= 1 + \frac{\sum_{i=1}^{d}\gamma_i i^2 - s}{\sum_{i=1}^{d}\beta_i i^2}.\end{aligned}$$

In this situation, if we take

$$\beta_i = \begin{cases}-\gamma_i & \text{if } \gamma_i < 0\\ 0 & \text{if } \gamma_i \geq 0\end{cases}, \alpha_i = \begin{cases}0 & \text{if } \gamma_i < 0\\ \gamma_i & \text{if } \gamma_i \geq 0\end{cases}, s = 0$$

then D is maximized for fix numbers $\boldsymbol{b}$, and the value D is

$$-\frac{\sum_{i\in\gamma^+}\gamma_i i^2}{\sum_{i\in\gamma^-}\gamma_i i^2}.$$

Put

$$\begin{aligned}\Delta(\boldsymbol{b}) &:= -\frac{\sum_{i\in\gamma^+}\gamma_i i^2}{\sum_{i\in\gamma^-}\gamma_i i^2}\\ &= -\frac{\sum_{i\in\gamma^+} i^2 \sum_{h=1}^{d} b_h v_i(\theta_h)}{\sum_{i\in\gamma^-} i^2 \sum_{h=1}^{d} b_h v_i(\theta_h)}.\end{aligned}$$

Theorem 3.8.

$$c_2(\Gamma)^2 \geq \sup\{\Delta(\boldsymbol{b}) \mid b_0 = 0, b_1, b_2, \ldots, b_d \geq 0\}.$$

For nonnegative numbers $\boldsymbol{b}$ and positive real c, we can check $\Delta(\boldsymbol{b}) = \Delta(c\boldsymbol{b})$. Therefore there exists $\boldsymbol{b}' \in [0,1]^{d+1}$ such that $\boldsymbol{b}'$ attain $\sup\{\Delta(\boldsymbol{b}) \mid b_0 = 0, b_1, b_2, \ldots, b_d \geq 0\}$.

Theorem 3.9 (Linial, London and Rabinovich[5]).

$$c_2(\Gamma)^2 \geq \max\{\Delta(\boldsymbol{b}) \mid b_0 = 0, b_1, b_2, \ldots, b_d \in [0,1]\}.$$

Lemma 3.10. *For any* $(\boldsymbol{a}, R) \in \mathcal{C} \cap \mathcal{S}_D$ *and* $(\boldsymbol{b}, s) \in \mathcal{C}^* \cap (-\mathcal{S}_D^*)$*, we have* $\sum_{h=0}^{d} a_h b_h + Rs = 0$.

Proof. Since $(\boldsymbol{a}, R) \in \mathcal{C}$ and $(\boldsymbol{b}, s) \in \mathcal{C}^*$, we have $(\boldsymbol{a}, R)\cdot(\boldsymbol{b}, s) \geq 0$. This implies $\sum_{h=0}^{d} a_h b_h + Rs \geq 0$. On the other hand, since $(\boldsymbol{a}, R) \in \mathcal{S}_D$ and $(\boldsymbol{b}, s) \in -\mathcal{S}_D^*$, we have $(\boldsymbol{a}, R)\cdot(\boldsymbol{b}, s) \leq 0$. This implies $\sum_{h=0}^{d} a_h b_h + Rs \leq 0$. Hence we get $\sum_{h=0}^{d} a_h b_h + Rs = 0$. □

Theorem 3.11 (Linial and Magen[6]). *Assume* $\boldsymbol{b}$ *satisfies* $\Delta(\boldsymbol{b}) = c_2(\Gamma)^2$ *and* $(\boldsymbol{a}, R) \in \mathcal{C} \cap \mathcal{S}_{c_2(\Gamma)^2}$. *Put* $\partial^2(i) = \frac{2}{n}\sum_{h=0}^{d} a_h(Q_h(0) - Q_h(i))$. *Then following hold.*

(*i*) $\sum_{h=0}^{d} a_h b_h = 0$.

(*ii*) *For* $i \in \gamma^+$, *the distance* i *is contracted, i.e.,* $\frac{\partial(i)}{i} = \min_{1\leq k\leq d}\frac{\partial(k)}{k}$.

(*iii*) *For* $i \in \gamma^-$, *the distance* i *is expanded, i.e.,* $\frac{\partial(i)}{i} = \max_{1\leq k\leq d}\frac{\partial(k)}{k}$.

Proof. The element of $\mathcal{C}^* \cap (-\mathcal{S}_{c_2(\Gamma)^2}^*)$ related to $\boldsymbol{b}$ forms

$$\left(\left\{b_h + \sum_{i\in\gamma^+}\gamma_i\tfrac{2}{n}(Q_h(0) - Q_h(i))\right\}_{h=0}^{d}, -\sum_{i\in\gamma^+}\gamma_i i^2\right) \tag{16}$$

or

$$\left(\left\{\sum_{i\in\gamma^-}\gamma_i\tfrac{2}{n}(Q_h(0) - Q_h(i))\right\}_{h=0}^{d}, -c_2(\Gamma)^2\sum_{i\in\gamma^-}\gamma_i i^2\right). \tag{17}$$

[(i) and (ii)] Apply the relation $(\boldsymbol{a}, R)$ and (16) to Lemma 3.10, we get

$$\sum_{h=0}^{d} a_h \left(b_h + \sum_{i\in\gamma^+} \gamma_i \tfrac{2}{n}(Q_h(0)-Q_h(i)) \right) - R \sum_{i\in\gamma^+} \gamma_i i^2$$

is vanish, i.e.,

$$\sum_{h=0}^{d} a_h b_h + \sum_{i\in\gamma^+} \gamma_i \partial^2(i) - R \sum_{i\in\gamma^+} \gamma_i i^2 = 0.$$

Since $R = \min_{1\le k\le d} \frac{\partial^2(k)}{k^2}$, we have

$$\sum_{h=0}^{d} a_h b_h + \sum_{i\in\gamma^+} \gamma_i i^2 \left(\frac{\partial^2(i)}{i^2} - \min_{1\le k\le d} \frac{\partial^2(k)}{k^2} \right) = 0.$$

Since $\sum_{h=0}^{d} a_h b_h \ge 0$, $\gamma_i i^2 > 0$ for i with $\gamma_i > 0$, and $\frac{\partial^2(i)}{i^2} - \min_{1\le k\le d} \frac{\partial^2(k)}{k^2} \ge 0$, we have $\sum_{h=0}^{d} a_h b_h = 0$ and $\frac{\partial^2(i)}{i^2} = \min_{1\le k\le d} \frac{\partial^2(k)}{k^2}$ for $i \in \gamma^+$.

[(iii)] Apply the relation $(\boldsymbol{a}, R)$ and (17) to Lemma 3.10, we get

$$\sum_{h=0}^{d} a_h \left(\sum_{i\in\gamma^-} \gamma_i \frac{2}{n}(Q_h(0)-Q_h(i)) \right) - c_2(\Gamma)^2 R \sum_{i\in\gamma^-} \gamma_i i^2,$$

is vanish, i.e.,

$$\sum_{i\in\gamma^-} \gamma_i \partial^2(i) - R c_2(\Gamma)^2 \sum_{i\in\gamma^-} \gamma_i i^2 = 0.$$

Since $Rc_2(\Gamma)^2 = \max_{1\le k\le d} \frac{\partial^2(k)}{k^2}$, we have

$$\sum_{i\in\gamma^-} \gamma_i i^2 \left(\frac{\partial^2(i)}{i^2} - \max_{1\le k\le d} \frac{\partial^2(k)}{k^2} \right) = 0.$$

Since $\gamma_i i^2 < 0$ for $i \in \gamma^-$, and $\frac{\partial^2(i)}{i^2} \le \max_{1\le k\le d} \frac{\partial^2(k)}{k^2}$, we have $\frac{\partial^2(i)}{i^2} = \max_{1\le k\le d} \frac{\partial^2(k)}{k^2}$ for $i \in \gamma^-$.

□

4 Upper and lower bound for $c_2(\Gamma)$

In this section, we give bounds for the Euclidean distortion of a distance-regular graph Γ.

4.1 Lower bound for $c_2(\Gamma)$

For $l \in \{2, 3, \dots, d\}$, let

$$m(l) := \min_{\substack{1\le j\le d,\\ Q_j(0)\ne Q_j(l)}} \frac{Q_j(0)-Q_j(1)}{Q_j(0)-Q_j(l)}$$

and

$$b_h^{(l)} = (Q_h(0)-Q_h(1)) - m(l)(Q_h(0)-Q_h(l))$$

$(1 \le h \le d)$. By the definition, $b_h^{(l)} \ge 0$ for $1 \le h \le d$. Then

$$\begin{aligned}\gamma_i =& \frac{1}{2}\sum_{h=1}^{d} b_h^{(l)} v_i(\theta_h)\\ =& \frac{1}{2}\sum_{h=1}^{d} (Q_h(0)-Q_h(1)) v_i(\theta_h)\\ & - m(l)(Q_h(0)-Q_h(l)) v_i(\theta_h)\\ =& -\frac{n}{2}\left(\delta_{1,i} - m(l)\delta_{l,i}\right),\end{aligned}$$

i.e., $\gamma_1 = -\frac{n}{2} < 0$, $\gamma_l = \frac{n}{2} m(l) > 0$ and $\gamma_i = 0$ for $i \ne 1, l$. Hence we have $\Delta((b_h^{(l)})_{h=1}^{d}) = l^2 m(l)$. Moreover we have the following result.

Theorem 4.1.

$$c_2(\Gamma)^2 \ge \max_{2\le l\le d} \min_{\substack{1\le j\le d,\\ Q_j(0)\ne Q_j(l)}} l^2 \frac{Q_j(0)-Q_j(1)}{Q_j(0)-Q_j(l)}.$$

By (8), Theorem 4.1 can be expressed by

$$c_2(\Gamma)^2 \ge \max_{2\le l\le d} \min_{1\le j\le d} \frac{l^2 v_l(\theta_0)}{v_1(\theta_0)} \frac{v_1(\theta_0)-v_1(\theta_j)}{v_l(\theta_0)-v_l(\theta_j)}.$$

4.2 Upper bound for $c_2(\Gamma)$

We consider the embedding F_j in terms of E_j. By Lemma 2.7, the distortion of F_j is

$$\frac{\frac{Q_j(0)-Q_j(1)}{1^2}}{\min_{\substack{2\le l\le d,\\ Q_j(0)\ne Q_j(l)}} \frac{Q_j(0)-Q_j(l)}{l^2}} = \max_{\substack{2\le l\le d,\\ Q_j(0)\ne Q_j(l)}} l^2 \frac{Q_j(0)-Q_j(1)}{Q_j(0)-Q_j(l)}.$$

Therefore we have the following result.

Theorem 4.2.

$$c_2(\Gamma)^2 \le \min_{1\le j\le d} \max_{\substack{2\le l\le d,\\ Q_j(0)\ne Q_j(l)}} l^2 \frac{Q_j(0)-Q_j(1)}{Q_j(0)-Q_j(l)}.$$

By (8), Theorem 4.2 can be expressed by

$$c_2(\Gamma)^2 \ge \min_{1\le j\le d} \max_{2\le l\le d} \frac{l^2 v_l(\theta_0)}{v_1(\theta_0)} \frac{v_1(\theta_0)-v_1(\theta_j)}{v_l(\theta_0)-v_l(\theta_j)}.$$

Remark 4.3. *Every distance-regular graph which we have already checked satisfies $j = 1$. Moreover almost distance-regular graphs satisfy $l = d$. However there exist some counterexamples of $l = d$. A distance-regular graph with*

$$\{22, 21, 20, 3, 2, 1; 1, 2, 3, 20, 21, 22\}$$

is such an example.

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