# 準離散加群と lifting 加群について

倉富要輔

# On quasi-discrete modules and lifting modules

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A module $M$ is said to be *lifting* if, it satisfies the following property: For any submodule $X$ of $M$, there exists a decomposition $M = X^* \oplus X^{**}$ such that $X^* \subseteq X$ and $X \cap X^{**}$ is a small submodule of $X^{**}$. This concept is dual one of extending modules. In the study of lifting modules, the following fundermental problem remains as an open problem :

"When is a direct sum $\oplus_I M_i$ of lifting modules $\{M_i\}_I$ lifting ?"

Let $M = \oplus_I M_i$ is extending. Then, for any submodule $X$ of $M = \oplus_I M_i$, we have a decomposition $M = X^* \oplus X^{**}$ such that $X$ is an essential submodule of $X^*$. In the dual problem above, "how $X^{**}$ should be" is an important point. From this viewpoint, in [6], we introduced "generalized injectivity (ojectivity)", a new concept of relative injectivity, and using this injectivity we gave some results for a finite direct sum of extending modules. Afterward, S.H.Mohamed and B.J.Müller [20] defined a dual to "generalized injectivity" as "generalized projectivity" and gave some results that concerned with the study of the problem above.

In this paper, we consider whether generalized projectivity passes to finite direct sums in the case that each module is hollow or quasi-discrete.

**Keywords:** Lifting modules; hollow module; quasi-discrete module;generalized projectivity.

## Introduction

A module $M$ is said to be *lifting* if, it satisfies the following property:

($D_1$) For any submodule $X$ of $M$, there exists a decomposition $M = X^* \oplus X^{**}$ such that $X^* \subseteq X$ and the kernel $X/X^*$ of the canonical epimorphism $M/X^* \to M/X$ is a small submodule of $M/X^*$, equivalently, $X \cap X^{**}$ is a small submodule of $X^{**}$. This concept is dual one of extending modules.

A ring $R$ is said to be (*semi*)*perfect* if any (cyclic) $R$-module has projective cover, equivalently, any (finitely generated) free $R$-module is lifting. These rings defined by H. Bass [2]. In 1963, E. Mares [18] defined 'semiperfect modules' as generalization of semiperfect rings as follows: A projective module $P$ is said to be *semiperfect* if any factor module of $P$ has a projective cover. Moreover, in 1983, through the study of dual concept '(quasi-)continuous modules', K. Oshiro [21] defined (quasi-)semiperfect modules as follows:

**Definition** A module $M$ is said to be *semiperfect* (resp. *quasi-semiperfect*) if $M$ is lifting with the following condition ($D_2$) (resp. ($D_3$)).

($D_2$) If $A$ is submodule of $M$ such that $M/A$ is isomorphic to a direct summand of $M$, then $A$ is a direct summand of $M$.

($D_3$) If $A$ and $B$ are direct summands of $M$ with $M = A + B$, then $A \cap B$ is a direct summand of $M$.

These modules are generalizations of semiperfect modules in the sense of Mares and were renamed as '(quasi-) discrete modules' by Mohamed and Müller [19].

The concept of projective cover is dual to the one of injective hull. However, for any module $M$, projective covers of $M$ need not exist. For this reason, any submodule of a module $M$ has a closure, but need not have a co-closure in $M$. For example, a submodule $2\mathbb{Z}$ of $\mathbb{Z}_\mathbb{Z}$ does not have a co-closure in $\mathbb{Z}_\mathbb{Z}$. In such a meaning, it is hard to treat lifting modules more than extending modules.

Since extending and lifting properties of modules take roots inside of ring theory, over the past few decades a considerable number of studies have been made on these properties. However the following fundermental problems remain as open problems:

**Problem A** : When is a direct sum of extending modules extending ?

**Problem B** : When is a direct sum of lifting modules lifting ?

These problems are unsolved even in the case finite direct sums. Why are these problems difficult? The point of asserting on Problem A is the following:

For any submodule $X$ of $M = \oplus_I M_i$, assume that there exists a decomposition $M = X^* \oplus X^{**}$ such that $X$ is an essential submodule of $X^*$. Then, 'how $X^{**}$ should be' is important. We must find $X^{**}$ with the relation with each $M_i$. In many cases, we can take it with $\oplus_I M_i'$ as $X^*$, where each $M_i'$ is a submodule of $M_i$. However, there exists a example that this property does not hold (For example, $\mathbb{Z}$-module $G_{\mathbb{Z}} = \mathbb{Z} \oplus \mathbb{Z}$ is extending, but the property above does not hold). In 2002, K. Oshiro et.al. [6] asserted that extending modules are studied by dividing into 3 types below:

(A) For any submodule $X$ of $M$, there exists a direct summand $X^*$ of $M$ such that $X$ is essential submodule of $X^*$.

(B) Let $M = \oplus_I M_i$. For any submodule $X$ of $M$, there exist a direct summand $X^*$ of $M$ and submodules $M_i'$ of $M_i$ ($i \in I$) such that $M = X^* \oplus (\oplus_I M_i')$ and $X$ is essential submodule of $X^*$.

(C) For any submodule $X$ of $M$ and any decomposition $M = \oplus_J N_j$, there exist a direct summand $X^*$ of $M$ and submodules $N_j'$ of $N_j$ ($j \in J$) such that $M = X^* \oplus (\oplus_J N_j')$ and $X$ is essential submodule of $X^*$.

A module $M$ with the condition (A) is a usual extending module. A module $M$ with the condition (B) (resp. (C)) is said to be an *extending module for* $M = \oplus_I M_i$ (resp. *extending module with the internal exchange property*). In particular, a module $M$ is said to be *extending with the finite internal exchange property* if $M$ satisfies (C) for any finite index set $J$. A decomposition $M = \oplus_I M_i$ is said to be *exchange decomposition* if, for any direct summand $X$ of $M$, there exist submodules $M_i'$ of $M_i$ such that $M = X \oplus (\oplus_I M_i')$ (cf. [20]). Hence a module $M = \oplus_I M_i$ with condition (B) is extending with the exchange decomposition $M = \oplus_I M_i$.

In [6], Oshiro et.al. introduced the generalized relative injectivity as follows:

Let $A$ and $B$ be modules. $A$ is said to be *generalized $B$-injective* (or *$B$-ojective*) if, any submodule $X$ of $B$ and any homomorphism $f : X \to A$, there exist decompositions $A = A_1 \oplus A_2$, $B = B_1 \oplus B_2$, a homomorphism $h_1 : B_1 \to A_1$ and a monomorphism $h_2 : A_2 \to B_2$ satisfying the following properties (*), (**):

(*) $X \subseteq B_1 \oplus h_2(A_2)$.

(**) For $x \in X$, we express $x$ in $B = B_1 \oplus B_2$ as $x = x_1 + x_2$, where $x_1 \in B_1$ and $x_2 \in B_2$. Then $f(x) = h_1(x_1) + h_2^{-1}(x_2)$.

By using this relative injectivity, we obtain the following:

**Theorem A** [6, Theorem 2.15] Let $M_1, \cdots, M_n$ be extending modules and put $M = M_1 \oplus \cdots \oplus M_n$. Then the following conditions are equivalent:

(1) $M$ is extending for $M = M_1 \oplus \cdots \oplus M_n$.

(2) $M_i$ is generalized $\oplus_{j \neq i} M_j$-injective for any $i \in \{1, 2, \cdots, n\}$.

(3) $\oplus_{j \neq i} M_j$ is generalized $M_i$-injective for any $i \in \{1, 2, \cdots, n\}$.

In this paper, we introduce a dual concept of relative generalized injectivity and consider whether generalized projectivity passes to finite direct sums in the case that each module is hollow or quasi-discrete.

## 1. Generalized Projectivity and Direct sums of Lifting modules

Throughout this paper $R$ is a ring with identity and all modules considered are unitary right $R$-modules. A submodule $S$ of a module $M$ is said to be a *small* submodule, if $M \neq K + S$ for any proper submodule $K$ of $M$ and we write $S \ll M$ in this case. Let $M$ be a module and let $N$ and $K$ be submodules of $M$ with $K \subseteq N$. $K$ is said to be a *co-essential* submodule of $N$ in $M$ if $N/K \ll M/K$ and we write $K \subseteq_c N$ in $M$ in this case. Let $X$ be a submodule of $M$. $X$ is called *co-closed* submodule in $M$ if $X$ has not a proper co-essential submodule in $M$. $X'$ is called a *co-closure* of $X$ in $M$ if $X'$ is a co-closed submodule of $M$ with $X' \subseteq_c X$ in $M$. $K <_{\oplus} N$ means that $K$ is a direct summand of $N$. Let $M = M_1 \oplus M_2$ and let $\varphi : M_1 \to M_2$ be a homomorphism. Put $\langle M_1 \overset{\varphi}{\to} M_2 \rangle = \{m_1 - \varphi(m_1) \mid m_1 \in M_1\}$. Then this is a submodule of $M$ which is called *the graph* with respect to $M_1 \to M_2$. Note that $M = M_1 \oplus M_2 = \langle M_1 \overset{\varphi}{\to} M_2 \rangle \oplus M_2$.

The reader is referred to [3], [5], [8], [13], [19], [21]–[25] for research on lifting, (quasi-)discrete modules and exchange properties.

We recall Problem B.

**Problem B** : When is a direct sum of lifting modules lifting ?

In the same as extending modules, we should divide lifting modules in 3 types as follows:

(A') For any submodule $X$ of $M$, there exists a direct summand $X^*$ of $M$ such that $X^* \subseteq_c X$ in $M$.

(B') Let $M = \oplus_I M_i$. For any submodule $X$ of $M$, there exist a direct summand $X^*$ of $M$ and submodules $M_i'$ of $M_i$ ($i \in I$) such that $M = X^* \oplus (\oplus_I M_i')$ and $X^* \subseteq_c X$ in $M$.

(C') For any submodule $X$ of $M$ and any decompositon $M = \oplus_J N_j$ of $M$, there exist a direct summand $X^*$ of $M$ and submodules $N_j'$ of $N_j$ ($j \in J$) such that $M = X^* \oplus (\oplus_J N_j')$ and $X^* \subseteq_c X$ in $M$.

A module $M$ with the condition (A') is a usual lifting module. A module $M$ with the condition (B') is said to be *lifting for* $M = \oplus_I M_i$ (or *lifting with the exchange decomposition* $M = \oplus_I M_i$. A module $M$ with the condition (C') is said to be *lifting with the internal exchange property*. In particular, a module $M$ is said to be *lifting with the finite internal exchange property* if $M$ satisfies (C') for any finite index set $J$.

In 2000, Keskin [13] studied about this problem and obtained the following:

**Theorem 1.1** (Keskin) Let $M_1, \cdots, M_n$ be lifting modules and let $M = M_1 \oplus \cdots \oplus M_n$ be an amply supplemented module. If $M_i$ is $M_j$-projective ($i \neq j$), then $M$ is lifting.

In this theorem, converse implication need not hold. For example, $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ is lifting, but $\mathbb{Z}/2\mathbb{Z}$ is not $\mathbb{Z}/4\mathbb{Z}$-projective.

In 2004, Mohamed and Müller [20] studied the question whether the dual to Theorem A holds or not, and they defined a dual ojectivity as follows:

Let $A$ and $B$ be modules. $A$ is said to be *generalized $B$-projective* (or *$B$-dual ojective*) if, for any homomorphism $f : A \to X$ and any epimorphism $g : B \to X$, there exist decompositions $A = A_1 \oplus A_2$, $B = B_1 \oplus B_2$, a homomorphism $h_1 : A_1 \to B_1$ and an epimorphism $h_2 : B_2 \to A_2$ such that $g \circ h_1 = f|_{A_1}$ and $f \circ h_2 = g|_{B_2}$.

Now we introduce properties of generalized projectivity (cf.[14], [17], [20]).

**Proposition 1.2** Let $A$ and $B$ be modules. Then

(1) If $A$ is $B$-projective then $A$ is generalized $B$-projective.

(2) Let $B$ be a lifting module. If $A$ is generalized $B$-projective then $A$ is im-small $B$-projective.

(3) If $A$ is generalized $B$-projective, then $A$ is generalized $B'$-projective for any direct summand $B'$ of $B$.

(4) Let $A$ be a module with the finite internal exchange property. If $A$ is generalized $B$-projective, then $A'$ is generalized $B$-projective for any direct summand $A'$ of $A$.

(5) Let $M = A \oplus B$ be amply supplemented. If $A$ is generalized $B$-projective, then $A'$ is generalized $B$-projective for any direct summand $A'$ of $A$.

(6) Let $A$ be an indecomposable module. Then $A$ is generalized $B$-projective if and only if $A$ is almost $B$-projective (cf.[1], [9], [10]).

The following is due to Mohamed and Müller [20].

**Theorem 1.3** (Mohamed-Müller) Let $M = A \oplus B$. Then $A$ is generalized $B$-projective if and only if whenever $M = N + B$, we have $M = N' \oplus A' \oplus B' = N' + B$ with $N' \subseteq N$, $A' \subseteq A$ and $B' \subseteq B$.

In [14], we obtained the following:

**Theorem 1.4** (cf. [14]) Let $M_1, \cdots, M_n$ be lifting modules and put $M = M_1 \oplus \cdots \oplus M_n$. Then the following conditions are equivalent:

(1) $M$ is lifting for $M = M_1 \oplus \cdots \oplus M_n$.

(2) $A$ and $B$ are mutually generalized projective for any $A <_{\oplus} (\oplus_K M_k)$ and any $B <_{\oplus} (\oplus_L M_l)$, where $K$ and $L$ are any two disjoint nonempty subsets of $\{1, \cdots, n\}$.

(3) $M_i'$ and $T$ are relative generalized projective for any $M_i' <_{\oplus} M_i$ and any $T <_{\oplus} (\oplus_{j \neq i} M_j)$, where $i \in \{1, \cdots, n\}$.

**Theorem 1.5** (cf. [14]) Let $M_1, \cdots, M_n$ be lifting modules with the finite internal exchange property and put $M = M_1 \oplus \cdots \oplus M_n$. Then the following conditions are equivalent:

(1) $M$ is lifting with the finite internal exchange property.

(2) $M$ is lifting for $M = M_1 \oplus \cdots \oplus M_n$.

(3) $\oplus_K M_k$ and $\oplus_L M_l$ are relative generalized projective for any two nonempty disjoint subsets $K$ and $L$ of $\{1, \cdots, n\}$.

(4) $M_i$ and $\oplus_{j \neq i} M_j$ are relative generalized projective for any $i \in \{1, \cdots, n\}$.

The study of a direct sum of hollow modules (indecomposable lifting modules) have been researched by several authors, e.g., Baba-Harada[1], Dung[4], Harada[7], Inoue[11], Keskin[13] etc. Now we introduce the following:

**Lemma 1.6** Let $M_1, \cdots, M_n$ be lifting modules ($\oplus$-supplemented modules) and put $M = M_1 \oplus \cdots \oplus M_n$. Then, for any submodule $X$ of $M$, there exists a direct summand $M_i'$ of $M_i$ $(i = 1, \cdots, n)$ such that $M = X + (M_1' \oplus \cdots \oplus M_n')$ and $X \cap (M_1' \oplus \cdots \oplus M_n') \ll M_1' \oplus \cdots \oplus M_n'$.

By using results above, we see the following:

**Theorem 1.7** (cf. [15]) Let $H_1, \cdots, H_n$ be hollow modules and put $M = H_1 \oplus \cdots \oplus H_n$. Then the following conditions are equivalent:

(1) $M$ is lifting with the (finite) internal exchange property.

(2) $M$ is lifting for $M = H_1 \oplus \cdots \oplus H_n$.

(3) $H_i$ is generalized $H_j$-projective $(i \neq j)$.

*Proof.* Let $M$ be a module with indecomposable decomposition. Then $M$ has the finite internal exchange property if and only if $M$ has the internal exchange property. Hence, by Theorem 1.5, (1) $\Leftrightarrow$ (2) holds. (2) $\Rightarrow$ (3) holds by Proposition 1.2 and Theorem 1.4.

(3) $\Rightarrow$ (2) : By Theorem 1.5, we only need show the case of $n \geq 3$. Assume that the statement holds the case of $n-1$. Let $X$ be a proper submodule of $M$ and put $I = \{1, \cdots, n\}$. We may assume that $X$ is not small in $M$. By Lemma 1.6, there exists a nonempty subset $J$ of $I$ such that $M = X + (\oplus_J H_j)$ and $X \cap (\oplus_J H_j) \ll \oplus_J H_j$. Now we put $K = I - J$. Since $X$ is not small in $M$, $K$ is nonempty and so we can take $\alpha \in J$ and $\beta \in K$ .Let $\nu : M \to M/(X + (\oplus_{J-\{\alpha\}} H_j))$ be the canonical epimorphism and put $f = \nu|_{H_\beta}$ and $g = \nu|_{H_\alpha}$. Then $g$ is an epimorphism. Since $H_\beta$ is generalized $H_\alpha$-projective, there exists a homomorphism $\varphi : H_\beta \to H_\alpha$ such that $g \circ \varphi = f$ $\cdots (i)$ or there exists an epimorphism $\varphi : H_\alpha \to H_\beta$ such that $f \circ \varphi = g \cdots (ii)$.

The case $(i)$: For any $y \in \langle H_\beta \overset{\varphi}{\to} H_\alpha \rangle$, there exists $h_\beta \in H_\beta$ with $y = h_\beta - \varphi(h_\beta)$. Then $\nu(y) = \nu(h_\beta - \varphi(h_\beta)) = f(h_\beta) - g\varphi(h_\beta) = f(h_\beta) - f(h_\beta) = 0$. Hence $\langle H_\beta \overset{\varphi}{\to} H_\alpha \rangle \subseteq X + (\oplus_{J-\{\alpha\}} H_j)$. Now we put $Q = \langle H_\beta \overset{\varphi}{\to} H_\alpha \rangle \oplus (\oplus_{J-\{\alpha\}} H_j)$. Then

$$M = Q \oplus H_\alpha \oplus (\oplus_{K-\{\beta\}} H_k) \quad \text{and} \quad Q \subseteq X + (\oplus_{J-\{\alpha\}} H_j).$$

And we see $Q = (Q \cap X) + (\oplus_{J-\{\alpha\}} H_j)$ $\cdots (*)$. As h.dim$Q < n-1$, by assumption, there exists a decomposition $Q = Q' \oplus (\oplus_{J-\{\alpha\}} \overline{H_j}) \oplus \overline{\langle H_\beta \overset{\varphi}{\to} H_\alpha \rangle}$ such that $Q' \subseteq_c Q \cap X$ in $Q$, where $\overline{H_j} \subseteq H_j$ and $\overline{\langle H_\beta \overset{\varphi}{\to} H_\alpha \rangle} \subseteq \langle H_\beta \overset{\varphi}{\to} H_\alpha \rangle$. By $(*)$, we see $Q' \neq 0$ and so $\overline{\langle H_\beta \overset{\varphi}{\to} H_\alpha \rangle} = 0$ or $\overline{H_j} = 0$ for some $j \in J - \{\alpha\}$. And we see $M = Q' \oplus (\oplus_{J-\{\alpha\}} \overline{H_j}) \oplus \langle H_\beta \overset{\varphi}{\to} H_\alpha \rangle \oplus H_\alpha \oplus (\oplus_{K-\{\beta\}} H_k) = Q' \oplus (\oplus_{J-\{\alpha\}} \overline{H_j}) \oplus \overline{H_\beta} \oplus H_\alpha \oplus (\oplus_{K-\{\beta\}} H_k)$, where $\overline{H_\beta} \subseteq H_\beta$. Now put $T = (\oplus_{J-\{\alpha\}} \overline{H_j}) \oplus \overline{H_\beta} \oplus H_\alpha \oplus (\oplus_{K-\{\beta\}} H_k)$. By assumption, for $T \cap X \subseteq T$, there exists a decomposition $T = Z \oplus (\oplus_{J-\{\alpha\}} \overline{H_j'}) \oplus \overline{H_\beta'} \oplus H_\alpha' \oplus (\oplus_{K-\{\beta\}} H_k')$ such that $Z \subseteq_c T \cap X$ in $T$, where $H_i' \subseteq H_i$, $\overline{H_j'} \subseteq H_j$ and $\overline{H_\beta'} \subseteq H_\beta$. Then we see

$$M = Q' \oplus Z \oplus T' \quad \textit{and} \quad X = Q' \oplus (T \cap X),$$

where $T' = (\oplus_{J-\{\alpha\}} H_j') \oplus \overline{H_\beta'} \oplus H_\alpha' \oplus (\oplus_{K-\{\beta\}} H_k')$. As $X \cap T' \ll T'$, $Q' \oplus Z \subseteq_c X$ in $M$. Therefore $M$ is lifting for $M = H_1 \oplus \cdots \oplus H_n$.

The case $(ii)$ : By the same argument as in the case $(i)$, we see $M$ is lifting for $M = H_1 \oplus \cdots \oplus H_n$.

□

## 2. Generalized Projectivity of Quasi-Discrete Modules

Since the structure of generalized projectivity is complicated, we do not know whether generalized projectivity passes to finite direct sums even in the case that each module is lifting. However, by Theorem 1.5, 1.7, we see the following:

**Proposition 2.1** Let $H_1, \cdots, H_n$ be hollow modules. If $H_i$ is generalized $H_j$-projective $(i \neq j)$, then $H_i$ and $\oplus_{j \neq i} H_j$ are relative generalized projective.

The following is generalization of the proposition above.

**Proposition 2.2** (cf. [16]) (1) Let $N$ be a quasi-discrete module and let $M = M_1 \oplus \cdots \oplus M_n$ be lifting for $M = M_1 \oplus \cdots \oplus M_n$. If $M_i$ is generalized $N$-projective for any $i \in \{1, \cdots, n\}$, then $M$ is generalized $N$-projective.

(2) Let $M$ be a quasi-discrete module and let $N = N_1 \oplus \cdots \oplus N_m$ be lifting for $N = N_1 \oplus \cdots \oplus N_m$. If $N_i$ and $M$ are relative generalized projective for any $i \in \{1, \cdots, n\}$, then $M$ is generalized $N$-projective.

By Theorem 1.5 and Proposition 2.2, we obtain the following:

**Theorem 2.3** Let $M_1, \cdots, M_n$ be quasi-discrete modules and put $M = M_1 \oplus \cdots \oplus M_n$. Then the following conditions are equivalent:

(1) $M$ is lifting with the (finite) internal exchange property.

(2) $M$ is lifting for $M = M_1 \oplus \cdots \oplus M_n$.

(3) $M_i$ is generalized $M_j$-projective $(i \neq j)$.

Now we introduce the following:

**Proposition 2.4** Let $M = M_1 \oplus \cdots \oplus M_n$ be an exchange decomposition and let $M_i$ be $M_j$-projective $(i \neq j)$. Then $M$ satisfies the condition $(D_3)$ if and only if each $M_i$ satisfies the condition $(D_3)$.

*Proof.* By [25, 41.14(1)], 'Only if' part holds.

'If' part : We only need show the case of $n = 2$. Let $U$ and $V$ be direct summands of $M$ with $M = U + V$. Since $M = M_1 \oplus M_2$ is an exchange decomposition, there exist decompositons $M_i = M_i' \oplus M_i''$ $(i = 1, 2)$ such that $M = U \oplus M_1' \oplus M_2'$. Since $M_1$ and $M_2$ satisfy $(D_3)$, by [25, 41.14(4)], $M_i'$ is $M_i''$-projective $(i = 1, 2)$. As $M_i$ is $M_j$-projective $(i \neq j)$, $M_i'$ is $M_j''$-projective $(i \neq j)$. Hence $M_1' \oplus M_2'$ is $M_1'' \oplus M_2''$-projective. Let $\pi : M \to M/V$ be a canonical epimorphism and put $f = \pi|_{M_1' \oplus M_2'}$, $g = \pi|_U$. As $M = U + V$, $g$ is an epimorphism. By $U \simeq M_1'' \oplus M_2''$, $M_1' \oplus M_2'$ is $U$-projective, and so there exists a homomorphism $h : M_1' \oplus M_2' \to U$ with $g \circ h = f$. Hence we see

$$M = \langle M_1' \oplus M_2' \xrightarrow{h} U \rangle \oplus U \quad \text{and} \quad V \supseteq \langle M_1' \oplus M_2' \xrightarrow{h} U \rangle.$$

Thus $V = \langle M_1' \oplus M_2' \xrightarrow{h} U \rangle \oplus (U \cap V)$. and so $U \cap V <_{\oplus} M$. Therefore $M$ satisfies $(D_3)$. □

As immediate consequences of Theorem 2.3, Proposition 2.4 and [19, Lemma 4.23], we obtain the following:

**Theorem 2.5** (cf. [13]) Let $M_1, \cdots, M_n$ be quasi-discrete modules and put $M = M_1 \oplus \cdots \oplus M_n$. Then $M$ is quasi-discrete if and only if $M_i$ is $M_j$-projective $(i \neq j)$.

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