Uniform extending 加群について

倉富要輔

On uniform extending modules

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A module M is said to be *(u-)extending* or *(u-)CS* if, for any (uniform) submodule X of M, there exist a direct summand X^* of M such that X is essential in X^*. An extending property, along with lifting property of modules have been studied by many researchers since 1980. However, the following fundermental problem remain as open problems:

When is a direct sum of CS (lifting) modules CS (lifting) ?

In this paper, we consider the open problem 'When is an infinite direct sum of uniform modules CS ?'.

Keywords: (u-)extending, (u-)ojective, internal exchange property.

1. Introduction

Throughout this paper R is a ring with identity and all modules considered are unitary right R-modules.

A submodule X of a module M is said to be *essential* in M, if $X \cap Y \neq 0$ for any non-zero submodule Y of M and we write $X \subseteq_e M$ in this case. Y is called a *closed* submodule in M or a *closed submodule* of M if Y has no proper essential extensions inside M. Let $A \subseteq B \subseteq M$. B is said to be *closure* of A in M if B is closed in M and $A \subseteq_e B$. $K <_\oplus N$ means that K is a direct summand of N.

The following is well known:

Proposition 1.1. *Let A, B, C be modules with $A \subseteq B \subseteq C$. Then $A \subseteq_e B$ and $B \subseteq_e C$ if and only if $A \subseteq_e C$*

Let $\mathcal{L}(M)$ be the family of all submodules of module M and let $\mathcal{A}$ be a class which is closed under submodules, essential extentions and isomorphic images. For example, a family of uniform modules, or modules with the finite uniform dimension, or nonsingular modules is closed under submodules, essential extentions and isomorphic images.

Let $M = M_1 \oplus M_2$ and let $\varphi : M_1 \to M_2$ be a homomorphism. Put $\langle M_1 \xrightarrow{\varphi} M_2\rangle = \{m_1 - \varphi(m_1) \mid m_1 \in M_1\}$. Then this is a submodule of M which is called *the graph* with respect to $M_1 \xrightarrow{\varphi} M_2$. Note that $M = M_1 \oplus M_2 = \langle M_1 \xrightarrow{\varphi} M_2\rangle \oplus M_2$.

Let $\{M_i \mid i \in I\}$ be a family of modules. The direct sum decomposition $M = \oplus_I M_i$ is said to be *exchangeable* if, for any direct summand X of M, there exists $\overline{M_i} \subseteq M_i$ $(i \in I)$ such that $M = X \oplus (\oplus_I \overline{M_i})$. A module M is said to have the *(finite) internal exchange property* if, any (finite) direct sum decomposition $M = \oplus_I M_i$ is exchangeable.

Lemma 1.2. *(cf.* [3, Theorem 2.15], [7, Proposition 2.5]*) Let $M = A \oplus B$. Then M has the finite internal exchange property if and only if A and B have the finite internal exchange property and $M = A \oplus B$ is exchangeable.*

A module M is said to be *$(\mathcal{A}$-)extending* if, for any submodule X of M (with $X \in \mathcal{A}$), there exists a direct summand N of M such that X is essential in N. In particular, A module M is said to be *u-extending* if, M is $\mathcal{A}$-extending for a family $\mathcal{A}$ of uniform modules.

Let A and B be modules. A is said to be *($\mathcal{A}$-)essentially B-injective* if, for any submodule X of B (with $X \in \mathcal{A}$) and any homomorphism $f : X \to A$ with $\ker f \subseteq_e X$, there exists a homomorphism $g : B \to A$ such that $g|_X = f$.

By the quite similar proof of [2, pp.16-17] and [5, Theorem 1.7], we can obtain the following:

Proposition 1.3. *Let A, B, A_α and B_α $(\alpha \in I)$ be modules. Then*

(1) *If B is ($\mathcal{A}$-)essentially A-injective if and only if B is ($\mathcal{A}$-)essentially K-injective for any submodule K of A (with $K \in \mathcal{A}$).*

(2) *Let B be an ($\mathcal{A}$-)extending module. Then B is ($\mathcal{A}$-)essentially A-injective if and only if B is ($\mathcal{A}$-)essentially bR-injective for any $b \in B$ (with $bR \in \mathcal{A}$).*

(3) *$\prod_I B_\alpha$ is ($\mathcal{A}$-)essentially A-injective if and only if B_α is ($\mathcal{A}$-)essentially A-injective for any $\alpha \in I$.*

(4) *Let A_α ($\alpha \in I$) and B be modules. Then B is ($\mathcal{A}$-)essentially $\oplus_I A_\alpha$-injective if and only if B is ($\mathcal{A}$-)essentially A_α-injective for any $\alpha \in I$.*

Proposition 1.4. *Let A_α ($\alpha \in I$) and B be modules. Then $\oplus_I A_\alpha$ is ($\mathcal{A}$-)essentially B-injective if and only if A_α is ($\mathcal{A}$-)essentially B-injective for any $\alpha \in I$ and the following condition (A_2') ((A_2'')) holds:*

(A_2') : For every choice of $b \in B$ and $a_i \in A_{\alpha_i}$ for distinct $\alpha_i \in I$ ($i \in \mathbb{N}$) such that $(0 : a_i) \supseteq (0 : b)$ and $\cap_{i=1}^{\infty} \ker \varphi_i \subseteq_e bR$ for the canonical homomorphism $\varphi_i : bR \to a_iR$ ($br \mapsto a_ir$), the ascending sequence $\cap_{i \geq k}(0 : a_i)$ ($k \in \mathbb{N}$) becomes stationary.

(A_2'') : For every choice of $b \in B'$ for some submodule $B' \in \mathcal{A}$ of B and $a_i \in A_{\alpha_i}$ for distinct $\alpha_i \in I$ ($i \in \mathbb{N}$) such that $(0 : a_i) \supseteq (0 : b)$ and $\cap_{i=1}^{\infty} \ker \varphi_i \subseteq_e bR$ for the canonical homomorphism $\varphi_i : bR \to a_iR$ ($br \mapsto a_ir$), the ascending sequence $\cap_{i \geq k}(0 : a_i)$ ($k \in \mathbb{N}$) becomes stationary.

A module M is said to be ($\mathcal{A}$-)*N-ojective* if, for any submodule X of N (with $X \in \mathcal{A}$) and any homomorphism $f : X \to M$, there exist decompositions $N = N_1 \oplus N_2$, $M = M_1 \oplus M_2$, a homomorphism $g_1 : N_1 \to M_1$ and a monomorphism $g_2 : M_2 \to N_2$ satisfying the following condition (*):

(*) For any $x \in X$, we express x and $f(x)$ in $N = N_1 \oplus N_2$ and $M = M_1 \oplus M_2$ as $x = n_1 + n_2$ and $f(x) = m_1 + m_2$, respectively. Then $g_1(n_1) = m_1$ and $g_2(m_2) = n_2$.

For undefined terminologies, the reader is referred to [1], [2], [5] and [8].

2. Main Results

In this section, we consider the case that $\mathcal{A}$ is a class of uniform modules. First we define the following:

Definition Let $\mathcal{A}$ be a class of uniform modules. A module M is said to be *u-extending* if M is $\mathcal{A}$-extending. A module M is said to be *u-N-ojective* (*u-essentially N-injective*) if M is *$\mathcal{A}$-N-ojective* ($\mathcal{A}$-essentially N-injective).

Proposition 2.1. *Let N be a u-extending module. If M is u-N'-ojective for any indecomposable direct summand N' of N, then M is u-essentially N-injective.*

Proof. Let X be a uniform submodule of N and let f be a homomorphism from X to M with $\ker f \subseteq_e X$ (that is, $\ker f \neq 0$). Since N is u-extending, there exists a direct decomposition $N = N' \oplus N''$ with $X \subseteq_e N'$. As M is u-N'-ojective, we see that either there exists a homomorphism $g : N' \to M$ with $g|_X = f$ or there exists a monomorphism $h : M \to N'$ with $h^{-1}|_X = f$.

Since $\ker f \neq 0$ and h is a monomorphism with $h^{-1}|_X = f$, there does not exist $h : M \to N'$. Hence there exists a homomorphism $g : N' \to M$ with $g|_X = f$. Define $g^* : N = N' \oplus N'' \to M$ by $g^*(n' + n'') = g(n')$, where $n' \in N'$ and $n'' \in N''$. Then f is extend to g^*. □

Proposition 2.2. *Let $M_1, \cdots, M_n$ be u-extending modules with the finite internal exchange property and put $P = M_1 \oplus \cdots \oplus M_n$. Assume that P is u-extending and the decompositon $P = M_1 \oplus \cdots \oplus M_n$ is exchangeable. Then*

(1) *Any direct summand of P is u-extending and the decomposition $P = A_1 \oplus \cdots \oplus A_n \oplus B_1 \oplus \cdots \oplus B_n$ is exchangeable for any decomposition $M_i = A_i \oplus B_i$ ($i = 1, \cdots, n$).*

(2) *The decomposition $Q = A_1 \oplus \cdots \oplus A_n$ is exchangeable for any direct summand A_i of M_i ($i = 1, \cdots, n$).*

Proof. (1) By a similar proof of [3, Lemma 2.13].

(2) Let $M_i = A_i \oplus B_i$ ($i = 1, \cdots, n$) and let X be a direct summand of $Q = A_1 \oplus \cdots \oplus A_n$. By (1), there exists a decomposition $P = X \oplus (\oplus_{i=1}^{n} A_i') \oplus (\oplus_{i=1}^{n} B_i')$ such that X is an essential submodule of X^*. Thus we get $Q = X \oplus (\oplus_{i=1}^{n} A_i') \oplus [(\oplus_{i=1}^{n} B_i') \cap Q] = X \oplus (\oplus_{i=1}^{n} A_i')$. Thus the decomposition $Q = A_1' \oplus \cdots \oplus A_n'$ is exchangeable. □

Theorem 2.3. *Let M_1, M_2 be u-extending modules with the finite internal exchange property and put $M = M_1 \oplus M_2$. Then the following conditions are equivalent:*

(1) *M is u-extending with the finite internal exchange property.*

(2) *M is u-extending and the decomposition $M = M_1 \oplus M_2$ is exchangeable.*

(3) *M_i' is u-M_j'-ojective for any direct summand M_i' of M_i and any uniform direct summand M_j' of M_j ($i \neq j$).*

Proof. (1) $\Leftrightarrow$ (2) follows by Lemma 1.2.

(2) $\Rightarrow$ (3) : By Proposition 2.2 (2), we may prove the following :

"Let M_1 be a uniform module and M_2 be a u-extending module and put $M = M_1 \oplus M_2$. If M is u-extending and the decomposition $M_1 \oplus M_2$ is exchangeable, then M_2 is u-M_1-ojective."

Let X be a uniform submodule of M_1 and let $f : X \to M_2$ be a homomorphism. By the assumption, for $\langle X \xrightarrow{f} M_2\rangle \subseteq M$, there exists a decomposition $M = Z \oplus M_1'' \oplus M_2''$ such that $\langle X \xrightarrow{f} M_2\rangle \subseteq_e Z$ and $M_i'' \subseteq M_i$. Put $M_i' = (Z \oplus M_j'') \cap M_i$ $(i \neq j)$ and then $M_i = M_i' \oplus M_i''$ $(i = 1, 2)$. Hence we see

$$M = M_1' \oplus M_2' \oplus M_1'' \oplus M_2'' = Z \oplus M_1'' \oplus M_2''$$

and so $M_1' \oplus M_2' \simeq \pi_Z(M_1') \oplus \pi_Z(M_2') = Z$, where $\pi_Z : M = Z \oplus M_1'' \oplus M_2'' \to Z$ is the projection. Since $\langle X \xrightarrow{f} M_2\rangle \simeq X$ is uniform and $\langle X \xrightarrow{f} M_2\rangle \subseteq_e Z$, Z is uniform and hence $M_1' = 0$ or $M_2' = 0$.

(I) In the case of $M_1' = 0$, we see $M = M_2' \oplus M_1 \oplus M_2'' = Z \oplus M_1 \oplus M_2''$. Let $z \in Z$ and express z in $M = M_2' \oplus M_1 \oplus M_2''$ as $z = m_2' + m_1 + m_2''$. By $z - m_1 = m_2' + m_2'' \in (Z \oplus M_1'') \cap M_2 = M_2'$, we see $z = m_2' + m_1$. As $M_2' \simeq \pi_Z(M_2') = Z$, $m_2' = 0$ implies $m_1 = 0$. By $\langle X \xrightarrow{f} M_2\rangle \subseteq_e Z$, $m_1 = 0$ implies $z = m_2' = 0$. Thus we can define a monomorphism $g : M_2' \to M_1$ by $g(p_{M_2'}(z)) = p_{M_1}(z)$, where $p_{M_2'} : M = M_2' \oplus M_1 \oplus M_2'' \to M_2'$ and $p_{M_1} : M = M_2' \oplus M_1 \oplus M_2'' \to M_1$ are the projections. Hence $\langle X \xrightarrow{f} M_2\rangle \subseteq_e Z = \langle M_2' \xrightarrow{g} M_1\rangle$ and so $g^{-1}|_X = f$.

(II) In the case of $M_2' = 0$, since $M_1 = M_1'$ is uniform, we see

$$M = M_1 \oplus M_2 = Z \oplus M_2.$$

Hence we can define a homomorphism $h : M_1 \to M_2$ by $h(p_{M_1}(z)) = p_{M_2}(z)$, where $z \in Z$ and $p_{M_i} : M \to M_i$ is the projections $(i = 1, 2)$. Then $h|_X = f$.

By (I), (II), M_2 is u-M_1-ojective.

(3) $\Rightarrow$ (2) : Let X be a uniform submodule of M and put $X_i = M_i \cap X$ $(i = 1, 2)$. Let $p_{M_i} : M \to M_i$ be the projections $(i = 1, 2)$.

(a) In the case of $X_1 \neq 0$, since X is uniform, we see that $X_1 \subseteq_e X$ and $X_2 = 0$. Since $X \simeq p_{M_1}(X)$ is uniform and M_1 is u-extending, there exists a decomposition $M_1 = M_1' \oplus M_1''$ such that $p_{M_1}(X) \subseteq_e M_1'$. As $X_2 = 0$, we can define a homomorphism $f : p_{M_1}(X) \to p_{M_2}(X)$ by $f(p_{M_1}(x)) = p_{M_2}(x)$. Then $\ker f = X_1 \subseteq_e p_{M_1}(X)$. By Proposition 2.1 and 1.3 (1), M_2 is u-essentially M_1'-injective and hence there exists a homomorphism $f^* : M_1' \to M_2$ with $f^*|_{p_{M_1}(X)} = f$. Thus we see

$$M = \langle M_1' \xrightarrow{f^*} M_2\rangle \oplus M_1'' \oplus M_2$$

and

$$X = \langle p_{M_1}(X) \xrightarrow{f} p_{M_2}(X)\rangle \subseteq_e \langle M_1' \xrightarrow{f^*} M_2\rangle.$$

(b) In the case of $X_2 \neq 0$. By the same argument of (a), we obtain that $M = \langle M_2' \to M_1\rangle \oplus M_1 \oplus M_2''$ and $X = \langle p_{M_2}(X) \to p_{M_1}(X)\rangle \subseteq_e \langle M_2' \to M_1\rangle$.

(c) In the case of $X_1 = X_2 = 0$, for any $x \in X$, express x in $M = M_1 \oplus M_2$ as $x = m_1 + m_2$. Then $m_i = 0$ implies $m_j = 0$ $(i, j = 1, 2)$. Thus we can define an isomorphism $f : p_{M_1}(X) \to p_{M_2}(X)$ by $f(p_{M_1}(x)) = p_{M_2}(x)$. As $X \simeq p_{M_i}(X)$ is uniform $(i = 1, 2)$, there exists a decomposition $M_i = M_i' \oplus M_i''$ with $p_{M_i}(X) \subseteq_e M_i'$ $(i = 1, 2)$. Since M_2' is u-M_1'-ojective and M_i' is uniform, there exists a monomorphism $g_1 : M_1' \to M_2'$ with $g_1|_{p_{M_1}(X)} = f$ or there exists a monomorphism $g_2 : M_2' \to M_1'$ with $g_2^{-1}|_{p_{M_1}(X)} = f$. Thus we see

$$M = \langle M_i' \xrightarrow{g_i} M_j'\rangle \oplus M_j' \oplus M_1'' \oplus M_2''$$

and

$$X \subseteq_e \langle M_i' \xrightarrow{g_i} M_j'\rangle.$$

By (a), (b) and (c), M is u-extending and the decomposition $M = M_1 \oplus M_2$ is exchangeable. □

Now we consider that a finite direct sum of u-extending modules.

Theorem 2.4. *Let $M_1, \cdots M_n$ be u-extending modules with the finite internal exchange property and put $M = M_1 \oplus \cdots \oplus M_n$. Then the following conditions are equivalent:*

(1) *M is u-extending with the finite internal exchange property.*

(2) *M is u-extending and the decomposition $M = M_1 \oplus \cdots \oplus M_n$ is exchangeable.*

(3) *M_i' is u-M_j'-ojective for any direct summand M_i' of M_i and any uniform direct summand M_j' of M_j $(i \neq j)$.*

Proof. By Lemma 1.2, (1) $\Leftrightarrow$ (2) holds.

(2) $\Rightarrow$ (3) follows by Proposition 2.2 and Theorem 2.3.

(3) $\Rightarrow$ (2) In the case $n = 3$. Let $X \subseteq M$ and put $X_i = M_i \cap X$ $(i = 1, 2, 3)$.

(I) If $X_i \neq 0$ for some $i \in \{1, 2, 3\}$, then $X_i \subseteq_e X$. Let $p_i : M = M_1 \oplus M_2 \oplus M_3 \to M_i$ be the projection $(i = 1, 2, 3)$. Then we can define a homomorphism $f : p_i(X) \to \oplus_{j\neq i} M_j$ by $f(p_i(x)) = (1_M - p_i)(x)$. Since $p_i(X) \simeq X$ is uniform, there exists a decomposition $M_i = M_i' \oplus M_i''$ with $p_i(X) \subseteq_e M_i'$. By $0 \neq X_i = \ker f \subseteq p_i(X)$, we see $\ker f \subseteq_e p_i(X)$. By Proposition 2.1 and 1.3, $\oplus_{j\neq i} M_j$ is u-essentially M_i'-injective and hence there exists a homomorphism $f^* : M_i' \to \oplus_{j\neq i} M_j$ with $f^*|_{p_i(X)} = f$. Thus we see

$$M = \langle M_i' \xrightarrow{f^*} \oplus_{j\neq i} M_j\rangle \oplus M_i'' \oplus (\oplus_{j\neq i} M_j)$$

and

$$X = \langle p_i(X) \xrightarrow{f} (1_M - p_i)(X)\rangle \subseteq_e \langle M_i' \xrightarrow{f^*} \oplus_{j\neq i} M_j\rangle.$$

(II) Let $X_i = 0$ for any $i \in \{1,2,3\}$. By $X_3 = 0$, we can define $\alpha : p_{M_1 \oplus M_2}(X) \to p_3(X)$ by $\alpha(p_{M_1 \oplus M_2}(x)) = p_3(x)$, where $p_{M_1 \oplus M_2} : M = M_1 \oplus M_2 \oplus M_3 \to M_1 \oplus M_2$ is the projection. Thus $X = \langle p_{M_1 \oplus M_2}(X) \xrightarrow{\alpha} p_3(X) \rangle$. By Theorem 2.3, $M_1 \oplus M_2$ is u-extending and the decomposition $M_1 \oplus M_2$ is exchangeable. As $X \simeq p_{M_1 \oplus M_2}(X)$ is uniform, there exists a decomposition $M_1 \oplus M_2 = Y \oplus M_1' \oplus M_2'$ such that $p_{M_1 \oplus M_2}(X) \subseteq_e Y$ and $M_i' \subseteq M_i$. Since Y is uniform, $M_1 \oplus M_2 = Y \oplus M_i' \oplus M_j$ $(i,j = 1,2)$ and so we see $Y \simeq M_i'' < \oplus M_i$. Hence M_3 is u-Y-ojective. Thus there exists a homomorphism $g : Y \to M_3$ with $g|_{p_{M_1 \oplus M_2}(X)} = f$ or there exist a direct summand M_3' of M_3 and a monomorphism $h : M_3' \to Y$ with $h^{-1}|_{p_{M_1 \oplus M_2}(X)} = f$. Hence we see that

$$M = \langle Y \xrightarrow{g} M_3 \rangle \oplus M_i' \oplus M_j \oplus M_3 \quad \text{and} \quad X \subseteq_e \langle Y \xrightarrow{g} M_3 \rangle$$

or

$$M = \langle M_3' \xrightarrow{h} Y \rangle \oplus M_1 \oplus M_2 \oplus M_3'' \quad \text{and} \quad X \subseteq_e \langle M_3' \xrightarrow{h} Y \rangle .$$

Therefore the statement holds for $n = 3$.

By the same argument above, we see that the statement holds for any n. □

Theorem 2.5. *Let $\{M_\alpha \mid \alpha \in \Lambda\}$ be a family of u-extending modules with the finite internal exchange property and put $P = \oplus_\Lambda M_\alpha$. Then the following conditions are equivalent:*

(1)P is u-extending and the decomposition $P = \oplus_\Lambda M_\alpha$.

(2) (a) M_α' is u-M_β'-ojective for any direct summand M_α' of M_α and any uniform direct summand M_β' of M_β $(\alpha \neq \beta)$.

(b) (A_2'') holds for all M_α and $\{M_\beta | \beta \neq \alpha, \beta \in \Lambda\}$.

Proof. (1) ⇒ (2) : (a) follows from Proposition 2.2 and Theorem 2.3.

(b) : By Proposition 1.4, we may show that " $\oplus_{\Lambda - \{\beta\}} M_\alpha$ is u-essentially M_β-injective ".

Let X be a uniform submodule of M_β and let $f : X \to \oplus_{\Lambda - \{\beta\}} M_\alpha$ be a homomorphism with $\ker f \subseteq_e X$. Since M_β is u-extending, there exists a decomposition $M_\beta = A_\beta \oplus B_\beta$ with $X \subseteq_e A_\beta$. Put $N = A_\beta \oplus (\oplus_{\Lambda - \{\beta\}} M_\alpha)$. Then N is u-extending and the decomposition $N = A_\beta \oplus (\oplus_{\Lambda - \{\beta\}} M_\alpha)$ is exchangeable by Proposition 2.2. Thus there exists a decomposition $N = Z \oplus A_\beta' \oplus (\oplus_{\Lambda - \{\beta\}} M_\alpha')$ such that $\langle X \xrightarrow{f} \oplus_{\Lambda - \{\beta\}} M_\alpha \rangle \subseteq_e Z$, $A_\beta = A_\beta' \oplus A_\beta''$ and $M_\alpha = M_\alpha' \oplus M_\alpha''$. Then we see

$$Z \simeq \pi(Z) = A_\beta'' \oplus (\oplus_{\Lambda - \{\beta\}} M_\alpha''),$$

where $\pi : N = [A_\beta' \oplus (\oplus_{\Lambda - \{\beta\}} M_\alpha')] \oplus [A_\beta'' \oplus (\oplus_{\Lambda - \{\beta\}} M_\alpha'')] \to A_\beta'' \oplus (\oplus_{\Lambda - \{\beta\}} M_\alpha'')$ is the projection. Since Z is uniform and $0 \neq \ker f \subseteq Z \cap A_\beta$, we obtain $\oplus_{\Lambda - \{\beta\}} M_\alpha'' = 0$ and so $N = A_\beta \oplus (\oplus_{\Lambda - \{\beta\}} M_\alpha) = Z \oplus (\oplus_{\Lambda - \{\beta\}} M_\alpha)$. Hence we see

$$Z = \langle A_\beta \xrightarrow{g} \oplus_{\Lambda - \{\beta\}} M_\alpha \rangle,$$

where $g : A_\beta = p_{A_\beta}(Z) \to (1_N - p_{A_\beta})(Z)$ is the canonical homomorphism. Then we see $g|_X = f$ and so $\oplus_{\Lambda - \{\beta\}} M_\alpha$ is u-essentially M_β-injective.

(2) ⇒ (1) : Let X be a uniform submodule of P. Let $0 \neq x \in X$. Then there exists a finite subset F of Λ such that $xR \subseteq \oplus_F M_i$. By Theorem 2.4, $\oplus_F M_i$ is u-extending and hence there exists a decomposition $\oplus_F M_i = Y \oplus M_k' \oplus (\oplus_{F - \{k\}} M_i)$ such that $xR \subseteq_e Y$ and $M_k = M_k' \oplus M_k''$. By $xR \subseteq_e X$ and $Y \subseteq_e X$, we can define a homomorphism $f : p_Y(X) \to M_k' \oplus (\oplus_{F - \{k\}} M_i) \oplus (\oplus_{\Lambda - F} M_\alpha)$ $(p_Y(x) \mapsto (1_P - p_Y)(x))$ with $\ker f = Y \cap X \subseteq_e p_Y(X)$, where $p_Y : P = Y \oplus M_k' \oplus (\oplus_{\Lambda - \{k\}} M_\alpha) \to Y$ is the projection.

By Proposition 1.3, 1.4 and Theorem 2.3, $M_k' \oplus (\oplus_{\Lambda - \{k\}} M_\alpha)$ is u-essentially Y-injective. Hence there exists a homomorphism $g : Y \to M_k' \oplus (\oplus_{\Lambda - \{k\}} M_\alpha)$ with $g|_{p_Y(X)} = f$. Thus we see

$$\begin{aligned} X &= \langle p_Y(X) \xrightarrow{f} (1_P - p_Y)(X) \rangle \\ &\subseteq_e \langle Y \xrightarrow{g} M_k' \oplus (\oplus_{\Lambda - \{k\}} M_\alpha) \rangle \end{aligned}$$

and

$$P = \langle Y \xrightarrow{g} M_k' \oplus (\oplus_{\Lambda - \{k\}} M_\alpha) \rangle \oplus M_k' \oplus (\oplus_{\Lambda - \{k\}} M_\alpha).$$

Therefore P is u-extending and the decomposition $P = \oplus_\Lambda M_\alpha$ is exchangeable. □

The following is due to Kado-Kuratomi-Oshiro [(4)]:

Lemma 2.6. *Let $\{U_\alpha \mid \alpha \in \Lambda\}$ be a family of uniform modules and put $P = \oplus_\Lambda U_\alpha$. Then the following conditions are equivalent:*

(1) P is extending with the finite internal exchange property.

(2) P is extending and the decomposition $P = \oplus_\Lambda U_\alpha$ is exchangeable.

(3) (a) U_α is U_β-ojective for any $\alpha \neq \beta \in \Lambda$.

(b) (A_2') holds for all U_α and $\{M_\beta \mid \beta \neq \alpha, \beta \in \Lambda\}$.

(c) There does not exist an infinite sequence of proper monomorphisms $\{f_k : U_{i_k} \to U_{i_{k+1}}\}_{k \in \mathbb{N}}$ with all $i_k \in \Lambda$ distinct.

Proof. By [(4, Theorem 2.5)]. □

Let N, M_α $(\alpha \in \Lambda)$ be uniform modules. Then we see that "M_α is N-ojective iff M_α is u-N-ojective" and "(A_2') iff (A_2'') for all N and $\{M_\alpha \mid \alpha \in \Lambda\}$". Thus, by Theorem 2.5 and Lemma 2.6, we obtain the following:

Theorem 2.7. *Let $\{U_\alpha \mid \alpha \in \Lambda\}$ be a family of uniform modules and put $P = \oplus_\Lambda U_\alpha$. Then the following conditions are equivalent:*

(1) P is extending with the finite internal exchange property.

(2) P is extending and the decomposition $P = \oplus_\Lambda U_\alpha$ is exchangeable.

(3) (a) P is u-extending and the decomposition $P = \oplus_\Lambda U_\alpha$ is exchangeable.

(b) There does not exist an infinite sequence of proper monomorphisms $\{f_k : U_{i_k} \to U_{i_{k+1}}\}_{k\in\mathbb{N}}$ with all $i_k \in \Lambda$ distinct.

(4) (a) P is u-extending and the decomposition $P = \oplus_\Lambda U_\alpha$ is exchangeable.

(b) P satisfies the condition (LSS).

(5) (a) P is u-extending and the decomposition $P = \oplus_\Lambda U_\alpha$ is exchangeable.

(b) $P = \oplus_\Lambda U_\alpha$ satisfies the condition $(lsTn)$.

Proof. By Theorem 2.5 and [4, Theorem 2.5 and 2.10]. □

文　献

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