Barnes–Wall 格子の最短ベクトルからなる有限集合の持つ性質について

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Properties of the finite sets consisted of the minimal vectors of the Barnes–Wall lattices

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In 1959, Barnes and Wall constructed a family of lattices of rank 2^d, $d > 0$. The lattices are called the Barnes–Wall lattices and denoted by BW_{2^d}. It is well known that BW_4 and BW_8 are isomorphic to the $\mathbb{D}_4$-lattice and the $\mathbb{E}_8$-lattice, respectively, and from this fact, the minimal vectors of these lattices are kissing configurations in the spheres $\mathbb{S}^3$ and $\mathbb{S}^7$, respectively. From these points of view, it is important to research the Barnes–Wall lattices.

In this paper, we give some properties of the minimal vectors of the Barnes–Wall lattices related to the theory of association schemes.

Key words: Barnes–Wall lattice; association scheme; design.

1 Introduction

Let $\mathbb{S}^{m-1}$ be the unit sphere centered at the origin in the m-dimensional Euclidean space $\mathbb{R}^m$, endowed with the standard inner product $x \cdot y = \sum_{i=1}^m x_i y_i$ for $x = (x_1, x_2, \ldots, x_m), y = (y_1, y_2, \ldots, y_m) \in \mathbb{R}^m$. We recall about notion of lattices. Let $\{b_1, b_2, \ldots, b_m\}$ be a basis of $\mathbb{R}^m$. A *lattice* L in $\mathbb{R}^m$ is the $\mathbb{Z}$-module of $\{b_1, b_2, \ldots, b_m\}$, i.e.,

$$L = \left\{ \sum_{i=1}^m a_i b_i \mid a_i \in \mathbb{Z} \right\}.$$

For a lattice L, we denote $\mu(L) = \min\{x \cdot x \mid x \in L,\ x \neq 0\}$ the minimal norm, and $X_L := \{x \in L \mid x \cdot x = \mu(L)\}$ the set of minimal vectors.

In a series of paper(4, 5, 6, 7, 15), Barnes, Bolt, Room and Wall investigated a family of lattices L_d in $\mathbb{R}^{2^d}$. The lattice L_d satisfies that the dual lattice $L'_d := \{x \in \mathbb{R}^{2^d} \mid x \cdot y \in \mathbb{Z},\ y \in L_d\}$ is geometrically similar to L_d with $L_d \subseteq L'_d$, and if $d \neq 3$, the automorphism groups $\mathrm{Aut}(L_d) = \mathrm{Aut}(L'_d)$ are subgroups of index 2 in the real Clifford group $\mathcal{C}_d$. When $d = 3$, L_3 and L'_3 are two versions of the root lattice $\mathbb{E}_8$, and $\mathrm{Aut}(L_3) \cap \mathrm{Aut}(L'_3)$ has index 270 in $\mathrm{Aut}(L_3)$ and index 2 in $\mathcal{C}_3$.

The lattices L_d and L'_d can be defined in terms of an orthonormal basis $b_0, b_1, \ldots, b_{2^d-1}$ of $\mathbb{R}^{2^d}$ as follows. Let V be a d-dimensional vector space $\mathbb{F}_2^d$ over the finite field $\mathbb{F}_2$ and index the basis elements $b_0, b_1, \ldots, b_{2^d-1}$ by the elements of V. For each affine subspace $U \subseteq V$, let $\chi_U \in \mathbb{R}^{2^d}$ correspond to the characteristic function of U: $\chi_U := \sum_{i=1}^{2^d} \epsilon_i b_i$, where $\epsilon_i = 1$ if i corresponds to an element of U and $\epsilon_i = 0$ otherwise. Then L_d (resp. L'_d) is $\mathbb{Z}$-spanned by the set

$$\{2^{\lfloor (d-l+\delta)/2 \rfloor} \chi_U \mid 0 \leq l \leq d,\ U \subset V,\ \dim = l\},$$

where $\delta = 1$ for L_d and $\delta = 0$ for L'_d. We call L_d the *Barnes–Wall lattice of rank* 2^d. We also use BW_{2^d} for the notion of Barnes–Wall lattice of rank 2^d. Originally, the Barnes–Wall lattices are constructed by Barnes and Wall in 1959(4). These lattices are very interesting inasmuch they form one of the very few infinite families of lattices for which explicit computations can be made (density, kissing number, automorphism group etc.). It is well known that $\mathrm{BW}_4 = L_2$ and $\mathrm{BW}_8 = L_3$ are isomorphic to the D_4-root lattice and the E_8-root lattice, respectively, and from this fact, the minimal vectors of these lattices are kissing configurations in the spheres $\mathbb{S}^3$ and $\mathbb{S}^7$, respectively. Although they do not provide, in dimension ≥ 32, the best known lattice packings. Also the Barnes–Wall lattices have been studied in the theory of vertex operator algebras. From these points of view, it is important to research the Barnes–Wall lattices.

Extending scalars, we define the $\mathbb{Z}[\sqrt{2}]$-lattice

$$M_d := \sqrt{2} L'_d + L_d,$$

which we call the *balanced Barnes–Wall lattice*.

Remark 1.1 (Nebe–Rains–Sloane(14)). *For $d > 1$, the lattice M_d is a tensor product:*

$$\begin{aligned} M_d &= M_{d-1} \otimes_{\mathbb{Z}[\sqrt{2}]} M_1 \\ &= \underbrace{M_1 \otimes_{\mathbb{Z}[\sqrt{2}]} M_1 \otimes_{\mathbb{Z}[\sqrt{2}]} \cdots \otimes_{\mathbb{Z}[\sqrt{2}]} M_1}_{d\ times}. \end{aligned}$$

In view of Remark 1.1, we have the following simple and apparently new construction for the Barnes–Wall lattice L_d. Namely, L_d is the rational part of the $\mathbb{Z}[\sqrt{2}]$-lattice $M_1^{\otimes d}$, where M_1 is the $\mathbb{Z}[\sqrt{2}]$-lattice with Gram matrix $\begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix}$. For more about this construction see Nebe–Rains–Sloane(13).

Some properties of minimal vectors of the Barnes–Wall lattices have been investigated. We remark that the minimal norm of the Barnes–Wall lattice L_d is $\mu(L_d) = 2^{\lfloor d/2 \rfloor}$. Let X_d be the set of minimal vectors of the Barnes–Wall lattice L_d, i.e., $X_d = X_{L_d}$. Then it is known that the cardinality of X_d is $\#X_d = 2^d \prod_{i=1}^{d}(2^i+1)$. Also the detail of the inner product set of X_d is already known. Recall that, for a nonempty finite set $X \subset \mathbb{R}^m$, the *inner product set* of X is given by $A(X) := \{x \cdot y \mid x, y \in X\}$. Then the inner product set of X_d forms

$$A(X_d) = \{0, \pm 1, \pm 2, \pm 2^2, \ldots, \pm 2^{\lfloor d/2 \rfloor}\}.$$

In this paper, we give some properties of the minimal vectors of the Barnes–Wall lattices of general rank 2^d related to the theory of association schemes. The following are the main theorem in this paper.

Theorem 1.2. *the set X_d of minimal vectors of the Barnes–Wall lattice L_d has the structure of symmetric association scheme with the partition of $X_d \times X_d$ related to the inner product set $A(X_d)$.*

The notion of association scheme is refer to Section 2, and in Section 3, we give a proof of Theorem 1.2. In section 4, we mention relations between designs and X_d. Finally, we give some remarks about the 16-dimensional Barnes–Wall lattice $L_4 = \mathrm{BW}_{16}$ in section 5.

2 Association scheme

We begin with a review of basic definitions concerning association schemes. The reader is referred to Bannai–Ito[2] for the background material.

Definition 2.1. *A symmetric association scheme $\mathfrak{X} = (X, \{R_i\}_{i=0}^r)$ of class r consists of a finite set X and a set $\{R_i\}_{i=0}^r$ of non-empty binary relations on X satisfying:*

(*i*) $R_0 = \{(x,x) \mid x \in X\}$,

(*ii*) $\{R_i\}_{i=0}^r$ *is a partition of* $X \times X$,

(*iii*) ${}^tR_i = R_i$ *for each* $i \in \{0,1,\ldots,r\}$, *where* ${}^tR_i = \{(y,x) \mid (x,y) \in R_i\}$,

(*iv*) *the numbers*

$$\#\{z \in X \mid (x,z) \in R_i \textit{ and } (z,y) \in R_j\}$$

are constant whenever $(x,y) \in R_k$, for each $i,j,k \in \{0,1,\ldots,r\}$.

The numbers

$$\#\{z \in X \mid (x,z) \in R_i \text{ and } (z,y) \in R_j\}$$

are called the *intersection numbers* and denoted by $p_{i,j}^k$.

The following example of association schemes is related to finite group theory.

Example 2.2. *Let G be a transitive permutation group on Ω with the cardinality n. Let G acts on $\Omega \times \Omega$ in such a way that $g(x,y) := (gx, gy)$ for $x, y \in \Omega$, $g \in G$. Let $R_0, R_1, \ldots, R_r$ be the orbits of G on $\Omega \times \Omega$, where $R_0 = \{(x,x) \mid x \in X\}$. The R_i's are called the* orbitals *of G on $\Omega \times \Omega$. Then $\mathfrak{X} = (X, \{R_i\}_{i=0}^r)$ is an association scheme of class r.*

We remark that this association scheme may be not usually symmetry. Nevertheless, if G has "nice" properties, then the association scheme obtained from G is symmetry.

Let $M_X(\mathbb{R})$ denote the algebra of matrices over the real field $\mathbb{R}$ with rows and columns indexed by X. For $i \in \{0,1,\ldots,r\}$, the i-th *adjacency matrix* A_i in $M_X(\mathbb{R})$ of $\mathfrak{X}$ is defined by

$$(A_i)_{x,y} = \begin{cases} 1 & \text{if } (x,y) \in R_i, \\ 0 & \text{otherwise.} \end{cases}$$

From the definition of association schemes, it follows that

(A1) $A_0 = I$, where I is the identity matrix,

(A2) $A_0 + A_1 + \cdots + A_r = J$, where J is the all-one matrix, and $A_i \circ A_j = \delta_{i,j} A_i$ for $i, j \in \{0,1,\ldots,r\}$, where $\circ$ denotes the Hadamard product, that is, the entry-wise matrix product,

(A3) ${}^tA_i = A_i$ for each $i \in \{0,1,\ldots,r\}$,

(A4) $A_i A_j = \sum_{k=0}^r p_{i,j}^k A_k$, for each $i, j \in \{0,1,\ldots,r\}$.

The vector space $\mathfrak{A} = \mathrm{Span}_{\mathbb{R}}\{A_0, A_1, \ldots, A_r\}$ with a basis $\{A_i\}_{i=0}^r$ forms a commutative algebra and is called the *Bose–Mesner algebra* of $\mathfrak{X}$. It is well known that $\mathfrak{A}$ is semi-simple, hence $\mathfrak{A}$ has a second basis $E_0, E_1, \ldots, E_r$ satisfying the following conditions:

(E1) $E_0 = \frac{1}{|X|} J$,

(E2) $E_0 + E_1 + \cdots + E_r = I$ and $E_i E_j = \delta_{i,j} E_i$,

(E3) ${}^tE_i = E_i$ for each $i \in \{0,1,\ldots,r\}$,

(E4) $E_i \circ E_j = \frac{1}{|X|} \sum_{k=0}^r q_{i,j}^k E_k$ for some real numbers $q_{i,j}^k$, for each $i, j \in \{0,1,\ldots,r\}$.

Then $E_0, E_1, \ldots, E_r$ are the primitive idempotents of the Bose–Mesner algebra $\mathfrak{A}$. The *first eigenmatrix* $P = (P_i(j))_{j,i=0}^r$ and the *second eigenmatrix* $Q = (Q_i(j))_{j,i=0}^r$ of $\mathfrak{X}$ are defined by

$$A_i = \sum_{j=0}^r P_i(j)E_j \text{ and } E_i = \frac{1}{|X|}\sum_{j=0}^r Q_i(j)A_j,$$

respectively.

We call $\mathfrak{X}$ a *P-polynomial scheme* (or a *metric scheme*) with respect to the ordering $\{A_i\}_{i=0}^r$, if for each $i \in \{0, 1, \ldots, r\}$, there exists a polynomial v_i of degree i, such that $A_i = v_i(A_1)$. Moreover $\mathfrak{X}$ is called a P-polynomial scheme with respect to A_1 if it has the P-polynomial property with respect to some ordering $A_0, A_1, A_{i_2}, A_{i_3}, \ldots, A_{i_r}$. Dually, $\mathfrak{X}$ is called a *Q-polynomial scheme* (or a *cometric scheme*) with respect to the ordering $\{E_i\}_{i=0}^r$, if for each $i \in \{0, 1, \ldots, r\}$, there exists a polynomial v_i^* of degree i, such that $|X|E_i = v_i^*((|X|E_1)^\circ)$, where, for $f \in \mathbb{R}[t]$ and $M = (M_{x,y})_{x,y\in X} \in M_X(\mathbb{R})$, we define $f(M^\circ) = (f(M_{x,y}))_{x,y\in X}$. Moreover $\mathfrak{X}$ is called a Q-polynomial scheme with respect to E_1 if it has the Q-polynomial property with respect to some ordering $E_0, E_1, E_{i_2}, E_{i_3}, \ldots, E_{i_r}$. In fact, an ordering of a Q-polynomial association scheme with respect to E_1 is uniquely determined (cf. Kurihara–Nozaki(12)). It is known that v_i and v_i^* form systems of orthogonal polynomials(2).

3 Proof of Theorem 1.2

In this section, we prove Theorem 1.2 using group theory.

In order to prove Theorem 1.2, we introduce the notion of an automorphism group of the Barnes–Wall lattice L_d. An *automorphism group* $\mathrm{Aut}(L_d)$ of L_d is a subgroup of the orthogonal group $O(2^d, \mathbb{R})$ that preserves L_d.

Fact 3.1 (Griess(10)). *An automorphism group* $\mathrm{Aut}(L_d)$ *of* L_d *is isomorphic to* $2_+^{1+2d}\Omega^+(2d, 2)$.

Here are some fairly standard notations used for particular extensions of groups: p^k means an elementary abelian p-group; $p^{a+b+\cdots}$ means an iterated group extension, with factors $p^a, p^b, \ldots$ (listed in upward sense). The automorphism group has order

$$\begin{cases} 696729600 & \text{for } d = 3 \\ 2^{d^2+d+1}(2^d - 1)\prod_{i=1}^{d-1}(2^{2^i} - 1) & \text{otherwise,} \end{cases}$$

In this paper, we do not mention more details of the group $2_+^{1+2d}\Omega^+(2d, 2)$. For more details of $2_+^{1+2d}\Omega^+(2d, 2)$, see Griess(10), etc.

$\mathrm{Aut}(L_d)$ also acts on the minimal vectors X_d of the Barnes–Wall lattice L_d. Moreover X_d is homogeneous for $\mathrm{Aut}(L_d)$, i.e.,

Fact 3.2 (Griess(10)). $\mathrm{Aut}(L_d)$ *acts on* X_d *transitively.*

Proof of Theorem 1.2. From Example 2.2 and Fact 3.2, X_d has the structure of an association scheme in terms of orbitals. In order to complete the proof of Theorem 1.2, we have to check the partition obtained from orbitals coincides with the partition obtained from the inner product set $A(X_d)$. Let $\{R_i\}_{i=0}^r$ be the orbitals of $\mathrm{Aut}(L_d)$ on $X_d \times X_d$, and for $\alpha \in A(X_d)$, let $S_\alpha := \{(x, y) \in X_d \times X_d \mid x \cdot y = \alpha\}$. Now we begin to prove that for each R_i, there exists $\alpha \in A(X_d)$ such that $R_i = S_\alpha$. Since $Aut(L_d)$ is a subgroup of the orthogonal group $O(2^d, \mathbb{R})$, for each $g \in \mathrm{Aut}(L_d)$ and $x, y \in X_d$, we have $(gx) \cdot (gy) = x \cdot y$. Hence, for each R_i, there exists $\alpha \in A(X_d)$ such that $R_i \subseteq S_\alpha$. On the other hand, the number of the orbitals of $\mathrm{Aut}(L_d)$ on $X_d \times X_d$ is $3 + 2\lfloor d/2 \rfloor$, we have $\#\{R_i\} = \#A(X_d)$. This implies that for each R_i, there exists $\alpha \in A(X_d)$ such that $R_i = S_\alpha$. $\square$

4 Relations between designs and X_d

The concept of spherical designs was introduced by Delsarte–Goethals–Seidel(9), and we refer also to Bannai–Bannai(3) and(9) for detail of spherical designs.

Definition 4.1 (Spherical design). *Let t be a non-negative integer. A finite nonempty subset X of $\mathbb{S}^{m-1}$ is called a spherical t-design if*

$$\frac{1}{\nu(\mathbb{S}^{m-1})}\int_{\mathbb{S}^{m-1}} f(x)d\nu(x) = \frac{1}{|X|}\sum_{x\in X} f(x)$$

holds for all polynomials $f(x) = f(x_1, x_2, \ldots, x_m)$ of degree at most t. Here ν is the Lebesgue measure on $\mathbb{S}^{m-1}$.

Namely, a spherical design is a finite set of points on a sphere which is "well-distributed", in the sense that it allows numerical integration of functions on the sphere up to a certain accuracy. The *strength* of design X is defined by $t(X) := \max\{k \in \mathbb{Z} \mid X \text{ is a } k\text{-design but is not a } (k+1)\text{-design}\}$.

A finite subset X in $\mathbb{R}^m$ is called an *antipodal* if $-X \subseteq X$, where $-X := \{-x \mid x \in X\}$. By definition of design, the strength of an antipodal design must be odd.

Since the set X_d of minimal vectors of the Barnes–Wall lattice L_d is antipodal and from Bachoc[1], we can determine or estimate the strength of X_d for each $d \geq 1$:

Fact 4.2. *The following hold.*

(*i*) *When* $d = 1$, X_1 *is isomorphic to a square in* $\mathbb{S}^1$. *Namely this implies* $t(X_1) = 3$.

(*ii*) *When* $d = 2$, X_2 *is isomorphic to a* D_4*-root system in* $\mathbb{S}^3$. *Namely this implies* $t(X_2) = 5$.

(*iii*) *When this implies* $d = 3$, X_3 *is isomorphic to a* E_8*-root system in* $\mathbb{S}^7$. *Namely this implies* $t(X_3) = 7$.

(*iv*) *For* $d \geq 4$, $t(X_d) \geq 7$.

There are some relations between designs and Q-polynomial association schemes.

Fact 4.3 (cf.[3]). *Let* X *be a nonempty finite subset in* $\mathbb{S}^{m-1}$ *Let* $s(X) := \#A(X) - 1$. *Suppose* X *is antipodal and* $t(X) \geq 2s(X) - 3$, *then* X *has the structure of a* Q*-polynomial association scheme with relations obtained from the inner product set.*

From Facts 4.2 and 4.3, we obtain the Q-polynomial property of X_d.

Corollary 4.4. *The following hold.*

(*i*) *When* $d = 1$, $s(X_1) = 2$. *Thus* X_1 *is a* Q*-polynomial association scheme.*

(*ii*) *When* $d = 2$, $s(X_2) = 4$. *Thus* X_2 *is a* Q*-polynomial association scheme.*

(*iii*) *When* $d = 3$, $s(X_3) = 4$. *Thus* X_3 *is a* Q*-polynomial association scheme.*

Remark that when $d = 4$, X_4 does not satisfies $t(X) \geq 2s(X) - 3$. Thus we cannot check whether X_4 has the Q-polynomial property or not. In the next section, we will investigate the properties of the 16-dimensional Barnes–Wall lattice L_4 and its minimal vectors X_4.

5 The 16-dimensional Barnes–Wall lattice $L_4 = \mathrm{BW}_{16}$

This section due to Conway–Sloane[8]. There are several constructions of the16-dimensional Barnes–Wall lattice BW_{16}. Construction B applied to the first-order Reed–Muller code of length for which det = 256, minimal norm = 4, kissing number τ = 4320, the minimal vectors consist of 480 of the form $2^{-1/2}(\pm 2^2, 0^{14})$ and 3840 of the form $2^{-1/2}(\pm 1^8, 0^8)$, where the positions of the $\pm$ 1's form one of the 30 codewords of weight 8 in the first-order Reed-Muller code-and there are an even number of minus signs. The number of elements in each shell of BW_{16} is given in Table 1. That is, for each integer m, $N(m) := \#\{x \in \mathrm{BW}_{16} \mid x \cdot x^{1/2} = m\}$ appear in Table 1.

Table 1: The shells of BW_{16}.

m	$N(m)$	m	$N(m)$
0	1	32	8593797600
2	0	34	11585617920
4	4320	36	19590534240
6	61440	38	25239859200
8	522720	40	40979580480
10	2211840	42	50877235200
12	8960640	44	79783021440
14	23224320	46	96134307840
16	67154406	48	146902369920
18	135168000	50	172337725440
20	319809600	52	256900127040
22	550195200	54	295487692800
24	1147643520	56	431969276160
26	1771683840	58	487058227200
28	3371915520	60	699846624000
30	4826603520	62	776820326400

BW_{16} may be constructed from the Leech lattice Λ_{24}. There are involutory symmetries of Λ_{24} that fix a 16-space, and the portion of Λ_{24} that lies in such a space is a copy of BW_{16}.

As the previous facts for L_4, L_4 may be a variable lattice. From this reason, we also hope that X_4 has the Q-polynomial property like the cases for $d = 1, 2, 3$. Unfortunately, X_4 does not have the Q-polynomial property.

Theorem 5.1. *For* $\alpha \in A(X_4)$, *let* $R_\alpha := \{(x, y) \in X_4 \times X_4 \mid x \cdot y = \alpha\}$. *Then the association scheme* $(X_4, \{R_\alpha\}_{\alpha \in A(X_4)})$ *is not a* Q*-polynomial association scheme.*

In order to prove this theorem, we use an excess theorem for 2-designs. For a finite subset X in $\mathbb{S}^{m-1}$ of size n, the vector space of $\mathbb{R}$-valued functions on X is denoted by $C(X)$. We equip $C(X)$ with an inner product by

$$(f, g) = \frac{1}{n} \sum_{x \in X} f(x) g(x)$$

for $f, g \in C(X)$. For a polynomial $p \in \mathbb{R}[t]$ and $a \in X$, we define the *zonal polynomial* $\zeta_a(p) : X \to \mathbb{R}$ *of* p *at* a by $\zeta_a(p)(x) = p(a \cdot x)$. We further define the vector spaces $\mathrm{Pol}_k(X)$ recursively by setting:

- $\mathrm{Pol}_0(X)$ is the set of constant functions on X,

- $\mathrm{Pol}_1(X) = \mathrm{Span}_{\mathbb{R}}\{\zeta_a(p) \mid a \in X, \deg p \le 1\}$,
- $\mathrm{Pol}_k(X) = \mathrm{Span}_{\mathbb{R}}\{fg \mid f \in \mathrm{Pol}_1(X),\ g \in \mathrm{Pol}_{k-1}(X)\}$ for $k \ge 2$.

We set

$$S(X) := \min\{0 \le i \le s(X) \mid \mathrm{Pol}_i(X) = C(X)\},$$

and we call $S(X)$ the *degree* of X. It is easy to show $S(X) \le s(X)$. Let $\mathrm{Harm}_0(X) = \mathrm{Pol}_0(X)$ and we define

$$\mathrm{Harm}_k(X) = \mathrm{Pol}_k(X) \cap \mathrm{Pol}_{k-1}(X)^{\perp} \text{ for } k \ge 1.$$

Its elements are *harmonic polynomials of degree k*. From the definition of the degree of X, we get

- $\mathrm{Harm}_j(X) \ne \{0\}$ for $0 \le j \le S(X)$,
- $C(X) = \bigoplus_{i=0}^{S(X)} \mathrm{Harm}_i(X)$.

For $\alpha \in A(X)$, denote $\#R_\alpha$ by κ_α. We put $Z^*(t) = \prod_{\alpha \in A(X)}(t - \alpha)$. We define an inner product on $\mathbb{R}[t]/(Z^*)$ by, for $p, q \in \mathbb{R}[t]/(Z^*)$,

$$\langle p, q \rangle = \frac{1}{n^2} \sum_{\alpha \in A(X)} \kappa_\alpha p(\alpha) q(\alpha).$$

The *predegree polynomials* $q_0, q_1, \dots, q_{s(X)}$ of X are the polynomials satisfying $\deg q_k = k$ and $\langle q_k, q_h \rangle = \delta_{k,h} q_k(m)$ for any $k, h \in \{0, 1, \dots, s(X)\}$. As a sequence of orthogonal polynomials, the predegree polynomials satisfy a three-term recurrence of the form

$$t q_k = b^*_{k-1} q_{k-1} + a^*_k q_k + c^*_{k+1} q_{k+1} \quad (0 \le k \le s(X)),$$

where the constants b^*_{k-1}, a^*_k and c^*_{k+1} are the Fourier coefficients of tq_k in terms of $\{q_i\}_{i=0}^{s(X)}$ respectively (and $b^*_{-1} = c^*_{s(X)+1} = 0$).

Fact 5.2 (Excess theorem for 2-designs, cf. Kurihara[11]). *Suppose a spherical 2-design X is with $S(X) = s(X)$. Then the inequality*

$$\dim \mathrm{Harm}_{s(X)}(X) \le q_{s(X)}(m)$$

holds and equality is attained if and only if X has the structure of a Q-polynomial association scheme with the relation obtained from $A(X)$.

Proof of Theorem 5.1. Assume $(X_4, \{R_\alpha\}_{\alpha \in A(X_4)})$ is a Q-polynomial association scheme. By Fact 5.2, the last predegree polynomial q_6 of X_4 satisfies

$$q_6(16) = \dim \mathrm{Harm}_6(X_4)$$

This implies that $q_6(16)$ must be an integer.

From an easy calculation, we get κ_α's of X_4: $A(X_4) = \{0, \pm 1, \pm 2 \pm 4\}$ and $\kappa_0 = 1710 \cdot 4320$, $\kappa_{\pm 1} = 1024 \cdot 4320$, $\kappa_{\pm 2} = 280 \cdot 4320$ and $\kappa_{\pm 4} = 1 \cdot 4320$. Also we can determine the predegree polynomials of X_4:

- $q_0(t) = 1$,
- $q_1(t) = t$
- $q_2(t) = \dfrac{9}{16}(-16 + t^2)$
- $q_3(t) = \dfrac{15}{64}\left(-\dfrac{128}{3}t + t^3\right)$
- $q_4(t) = \dfrac{17}{6656}(8192 - 1152t^2 + 15t^4)$
- $q_5(t) = \dfrac{3}{1024}t(4096 - 160t^2 + t^4)$
- $q_6(t) = \dfrac{3(-917504 + 182272t^2 - 3760t^4 + 13t^6)}{212992}$

and we have $q_6(16) = 3192/13 \notin \mathbb{Z}$. This is a contradiction. Therefore $(X_4, \{R_\alpha\}_{\alpha \in A(X_4)})$ is not a Q-polynomial association scheme. □

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